Solutions 10

Question 1 The infinitesimal length scaling factor from P to z is $\frac{|z|^2+1}{2}$, that from z to w is $|\phi'(z)| = |-\bar{v}z+\bar{u}|^{-2}$ (the conformal map ϕ rotates tangent vectors at z by $\arg\phi'(z)$ and scales them by $|\phi'(z)|$, as can be seen from its 2×2 matrix form), and that from w to Q is $\frac{2}{|w|^2+1}$. So the composite scaling is

$$\begin{aligned}\frac{|z|^2+1}{2}\cdot\frac{1}{|-\bar{v}z+\bar{u}|^2}\cdot\frac{2}{|uz+v|^2|-\bar{v}z+\bar{u}|^{-2}+1} &= \frac{|z|^2+1}{|uz+v|^2+|-\bar{v}z+\bar{u}|^2}\\ &= \frac{|z|^2+1}{|u|^2|z|^2+|v|^2+uz\bar{v}+\bar{u}\bar{z}v+|v|^2|z|^2+|u|^2-\bar{v}zu-v\bar{z}\bar{u}}=1\end{aligned}$$

Question 2

$$\frac{az+b}{cz+d}=\frac{a}{d}z+\frac{b}{d}$$

if $c=0$, whereas if $c\neq 0$ then

$$\frac{az+b}{cz+d}=\frac{a}{c}\frac{z+bc/a}{z+cd/a}=\frac{a}{c}+\frac{(bc-cd)/a}{z+cd/a}$$

which is a atranslation, an inversion, a scaling and a translation again.

Question 3
Checking invariance under each of the maps in Q2 is straightforward.

Question 4
Shift the imaginary line to the right by 1, $z\mapsto w=z+1$. Apply inversion $w\mapsto u=1/w$; this takes the line $\mathrm{Re}(z)=1$ to the circle of radius 1/2 centered at 1/2. Subtract 1/2 to center the circle, and scale it by 2. The map we get is

$$z\mapsto z+1\mapsto\frac{1}{z+1}\mapsto\frac{1}{z+1}-\frac{1}{2}\mapsto\frac{2}{z+1}-1.$$

Luckily, we see that $1\mapsto 0$. We get any other map by post-composing this with a Möbius map preserving the unit circle and fixing the origin. We know that the maps preserving the unit circle have the form $z\mapsto\frac{uz+v}{\bar{v}z+\bar{u}}$, with $|u|^2-|v|^2=1$; the condition $0\mapsto 0$ is then $v=0$, so $|u|=1$ and the map $z\mapsto uz/\bar{u}$ is then a rotation; so the most general map as desired is

$$z\mapsto\frac{2e^{i\theta}}{z+1}-e^{i\theta}.$$

Question 5
The map is $z\mapsto\bar{z}^{-1}$. See Needham's book for a nice discussion of *inversion about the unit circle*: to find the image of $z\neq 0$, draw the ray from 0 to z and choose the point on it at distance $1/|z|$

from the origin. 0 and ∞ get interchanged. Of course this helps you solve Q7 as well.
The antipodal map is $z \mapsto -\bar{z}^{-1}$.

Question 6

The following observation will be used: a line meets a circle at right angles iff it passes through the center. (Easy by symmetry).
Call C_1, C_2 the original circles. Let L be the line joining their centers. We will look for a circle C which meets and is orthogonal to all of C_1, C_2 and L. (This circle will be centered on L.) Assuming we found C, we then send the two intersection points $C \cap L$ to 0 and ∞, respectively. This maps C and L to two perpendicular lines, and C_1 and C_2 to two circles that meet both of these lines at right angles. Hence, these circles are centered at 0 and we have found our map.
To find C, choose a Möbius map sending one intersection point $C_2 \cap L$ to ∞. Then, C_2 becomes a line L_2, and L a line L'. Now L_2 and L' are still perpendicular and the centre of the image C_1' of C_1 lies on L'. We seek a circle C' centered at $P = L_2 \cap L'$ (so that it is orthogonal to both) and orthogonal to C_1'. But this is easy to find, provided that P lies outside C_1': draw the tangent lines from P to C_1' and note that the points where they touch C_1' must be on C'; this gives you the radius.
Finally, map back your C' to obtain the desired C.
When does the construction fail? Precisely when P is in the *interior* of C_1'. That means that C_1' separates P and ∞. But P and ∞ where the images of the two crossings $C_2 \cap L$. They are separated by C_1' iff C_1 separates those two crossing points, which happens exactly when C_1 and C_2 meet.

Question 8

We write the map as a composition of two steps, $z \mapsto w = \frac{z+1}{z-1}$ followed by $w \mapsto w^2$.
Look at the first map. By the previous problem (or by checking), it preserves the real axis but sends the upper half-plane to the lower one. The imaginary axis goes to a circle orthogonal to the real axis and passing through the images of 0 and ∞, which are -1 and 1; so the imaginary axis maps to the unit circle. Finally, the right half-plane must go to one side of the unit circle, and as $1 \mapsto \infty$ this will be the *outside*.
Squaring sends the extended real axis to the non-negative half of the same, sends the unit circle to itself, and the outside of the unit circle to itself.