# Solutions 1

**Question 1**

The function is continuous and has limits $-\infty$ at $-\infty$ and $+\infty$ at $+\infty$ so there is always at least one real root. The derivative is $3x^2-3p$, positive when $p<0$ (and also when $p=0$ except possibly at one point) so the function is strictly increasing in that case and there is a unique real root. Note that $p<0$ implies $q^2>p^3$.

If $p>0$ the function is strictly increasing up to $-\sqrt{p}$, has a local maximum there, decreases strictly until $\sqrt{p}$ where it has a local minimum and is strictly increasing thereafter. We get 3 real roots precisely when the local maximum value is positive and the local minimum value negative. These values are $2p\sqrt{p}-2q$ and $-2p\sqrt{p}-2q$. Thus, for 3 solutions we must have $p\sqrt{p}>q$ and $p\sqrt{p}>-q$, which together are equivalent to $p^3>q^2$.

**Question 2**

Note that $(r_+r_-)^3 = q^2-\Delta = p^3$. We must have $r_+r_- = p$, because they are both real. Then,

$$\begin{aligned}(r_++r_-)^3-3p(r_++r_-)-2q &= r_+^3+r_-^3+3r_+r_-(r_++r_-)-3p(r_++r_-)-2q\\ &= 2q+3p(r_++r_-)-3p(r_++r_-)-2q=0\end{aligned}$$

For $\omega r_+ + \omega^2 r_-$, the product is still $p$ and we have similarly

$$\begin{aligned}(\omega r_++\omega^2r_-)^3-3p(\omega r_++\omega^2r_-)-2q &= r_+^3+r_-^3+3r_+r_-(\omega r_++\omega^2r_-)-3p(\omega r_++\omega^2r_-)-2q\\ &= 2q+3p(\omega r_++\omega^2r_-)-3p(\omega r_++\omega^2r_-)-2q=0\end{aligned}$$

When $\Delta<0$, $r_+$ is complex. But choosing $r_-=\bar{r}_+$ gives again $r_+r_-=p$, because the cube is again $p^3$ and $r_+r_-=|r_+|^2$ is real. The calculation then runs exactly as above.

**Question 3**

Write $f(z)=f_nz^n+f_{n-1}z^{n-1}+\cdots+f_1z+f_0=\sum_{k=0}^n f_kz^k$, with $f_k$ real. Then,

$$\begin{aligned}\overline{f(z)}=\overline{f_nz^n+f_{n-1}z^{n-1}+\cdots+f_1z+f_0} &= \overline{f_n}\cdot\overline{z^n}+\overline{f_{n-1}}\cdot\overline{z^{n-1}}+\cdots+\overline{f_1}\cdot\bar{z}+\overline{f_0}=\\ &= f_n\bar{z}^n+f_{n-1}\bar{z}^{n-1}+\cdots+f_1\bar{z}+f_0=f(\bar{z})\end{aligned}$$

because the coefficients $f_k$ are real. So if $f(\alpha)=0$ then $f(\bar{\alpha})=\overline{f(\alpha)}=0$.

**Question 4**

Let $z=\cos\theta+\mathrm{i}\sin\theta$ and choose $\cos\frac{\theta}{2}=\mathrm{i}\sin\frac{\theta}{2}$ for $z^{1/2}$. Then, multiplying numerator and denominator by $z^{-1/2}$ we get the identity

$$(1+\cos\theta+\cos2\theta+\cdots+\cos n\theta)+\mathrm{i}(\sin\theta+\cdots+\sin n\theta)=\frac{\cos(n+\frac{1}{2})\theta-\cos\frac{\theta}{2}+\mathrm{i}\sin(n+\frac{1}{2})\theta+\mathrm{i}\sin\frac{\theta}{2}}{2\mathrm{i}\sin\frac{\theta}{2}}$$

and we get the advertised identities by equating real and imaginary parts.

**Question 5**

We have

$$1 + \omega + \cdots + \omega^{n-1} = \frac{1-\omega^n}{1-\omega} = 0$$

by de Moivre's formula.

**Question 6**

Because $|\omega| = 1$, the parallelogram spanned by 1 and $\omega$ is a rhombus, so the diagonal $1+\omega$ bisects the angle $\frac{2\pi}{n}$. Also, the modulus $|1+\omega|$ is $2\cos\frac{\pi}{n}$, because the second diagonal creates four congruent right-angled triangles with angles $\pi/n$. Therefore,

$$(1+\omega)^k = 2^k \cos^k \frac{\pi}{n} \left( \cos\frac{k\pi}{n} + \mathrm{i}\sin\frac{k\pi}{n} \right)$$