Question 1

(a) The denominator vanishes at $z = -1$ with a double zero. The numerator is holomorphic and does not vanish at $z = -1$. so we have a double pole at $z = -1$ with residue $-e^{-1}$, the derivative of the numerator at $z = -1$.

(b) Holomorphy is clear everywhere except at $z = 0$. There, the numerator vanishes with a triple zero, the first term in the Taylor series being $z^3/6$; so we have a simple pole with residue $1/6$.

(c) The denominator vanishes with a double zero at all integer multiples of $2\pi i$. Of these, the numerator vanishes at $z = 0$ only. So we have a simple pole at the origin with resudue 1, and double poles at $2\pi in$, $n \in \mathbb{Z} \setminus \{0\}$. The residues are obtained from the starts of the Taylor expansions near $z = 2\pi ni$, in terms of $w = (z - 2\pi ni)$

$$\begin{aligned}
\frac{\sin z}{(e^z - 1)^2} &= \frac{\sin(2\pi ni) + \cos(2\pi ni)w + O(w^2)}{(w + w^2/2 + O(w^3))^2} \\
&= \frac{1}{w^2}(\sin(2\pi ni) + \cos(2\pi ni)w + O(w^2))(1 - w + O(w^2)) \\
&= \frac{\sin(2\pi ni)}{w^2} + \frac{1}{w}(\cos(2\pi ni) - \sin(2\pi ni)) + O(1)
\end{aligned}$$

so the residue is $\cos(2\pi ni) - \sin(2\pi ni) = \cosh(2\pi n) - i\sinh(2\pi n)$

Question 3

This is half of the integral on the real line, for which we can use a contour along the upper half-disk of radius R and let $R \to \infty$. The integrand will be $ze^{iaz}/(z^4 + 4)$, which decays roughly as R^{-3} on the upper half-circle, so that integral decays like R^{-2}, leaving only the real line contribution.

There are two poles enclosed by the contour, at $z = \pm 1 + i$, with residues $e^{ia-a}/8i$ and $-e^{-ia-a}/8i$, adding up to $\frac{1}{4}e^{-a}\sin a$. So the original real integral, half the imaginary part of the complex integral, is π times that, which is the answer given.

Question 4

Double the integral to 2π and convert to a complex integral on the unit circle C,

$$(-i) \cdot \oint_C \frac{z + z^{-1}}{4 + z + z^{-1}} z^{-1} dz$$

The integrand, $\frac{z^2+1}{z^3+4z^2+z}$, has simple poles at $z = 0$ and $-2 + \sqrt{3}$ inside the unit circle (the other pole is outside). The residues are 1 and $-2/\sqrt{3}$, so the integral is

$$(-i) \cdot \pi i \cdot (1 - 2/\sqrt{3}) = \pi(1 - 2/\sqrt{3})$$

Question 5

See Schaum, §5.2

Question 1

(a) There is a removable singularity at $z = 0$, as can be seen from the Taylor expansion of $\sin z$ by cancelling a factor of z. There are further singularitues at $z = \pi n$, $n \in \mathbb{Z} \setminus \{0\}$, where $\sin z$ has simple zeroes. The residues are $(-1)^n \pi n$, the limit as $z \to \pi n$ of $(z - \pi n)\frac{z}{\sin z}$.

(b) The isolated singularities are at $z = 1/2\pi i n$. There is a non-isolated singularity at $z = 0$. The residue at $1/2\pi i n$ is found by setting $z = 1/2\pi i n + w$, in which case

$$\exp(1/z) = \exp\left(\frac{1}{1/2\pi i n + w}\right) = \exp\left(\frac{2\pi i n}{1 + 2\pi i n w}\right)$$
$$= \exp\left(2\pi i n + 4\pi^2 n^2 w + O(w^2)\right) = 1 + 4\pi^2 n^2 w + O(w^2)$$

so the residue is $1/4\pi^2 n^2$.

Question 3

With $z = e^{i\theta}$, $d\theta = (-i)dz/z$ and the integral becomes one iver the unit circle C

$$\frac{-i}{2} \oint_C \frac{2 + z + z^{-1}}{13z + 6iz^2 - 6i} dz = \oint_C \frac{-z^2 - 2z - 1}{2z(6z^2 - 13iz - 6)} dz$$

Now $6z^2 - 13iz - 6 = (2z - 3i)(3z - 2i)$ so the zeroes of the denominaror are at $0, 2i/3$ and $3i/2$, of which only the first two lies inside C. The residue at 0 is $\frac{1}{12}$, at $2i/3$ it is $-\frac{1}{12} - \frac{i}{5}$, so the integral is $2\pi/5$.

Question 4

We integrate $z^2/(z^4+4)^2$ around a quarter-disk of radius R in the first quadrant. The circle integral vanishes in the limit, since the integrand decays roughly like R^{-6}. The imaginary axis integral, where $z = iy$ ranging from R to 0, equals $i \cdot \int_0^R y^2 \cdot (y^4 + 4)^{-2} dy$. There is one singularity in the quadrant, a double pole at $z = 1 + i$. So our answer for the integral is $\frac{2\pi i}{1+i}\rho$, with the residue ρ at $1 + i$.

To compute ρ, we use the derivative formula for the residue:

$$\frac{d}{dz}\left(\frac{z^2}{(z^2 + 2i)^2(z + 1 + i)^2}\right) =$$
$$= \frac{2z}{(z^2 + 2i)^2(z + 1 + i)^2} - \frac{4z^3}{(z^2 + 2i)^3(z + 1 + i)^2} - \frac{2z^2}{(z^2 + 2i)^2(z + 1 + i)^3}.$$

Evaluated at $z = 1 + i$,

$$\frac{i - 1}{64} + \frac{1 + i}{64i} + \frac{1}{64(1 + i)} = \frac{1 - i}{128}$$

so the value of the integral is $\pi/64$.