Math185 – Midterm 2

Thursday, November 9, 2006, 9:40-11:00
This is a closed book exam.
Please write clearly and explain your reasoning,
unless instructed otherwise

Question 1, 6 pts
Determine the nature of the singularities and find the residues there for the following functions:

(a) $\dfrac{(z-1)e^z}{(z+1)^2}$ (b) $\dfrac{z-\sin z}{z^4}$ (c) $\dfrac{\sin z}{(e^z-1)^2}$

Question 2, 8 pts
True or false? Circle the correct answer; no reason necessary.

T F The function $\sqrt{z}$ has an isolated singularity at $z = 0$

T F A harmonic function on $\mathbb{C}$ that is bounded below must be constant

T F If f is a holomorphic function on $\mathbb{C}$ and γ, Γ are two piecewise regular curves with the same endpoints, then $\int_\gamma f(z)dz = \int_\Gamma f(z)dz$

T F The residue of $f(z) = \sin(z)/z$ at $z = 0$ is zero

T F If the holomorphic function f is never zero in a domain D, then we can define a single-valued branch of $\log f$ without any branch cuts

T F If the function f defined on $\mathbb{C}^*$ is holomorphic and bounded, then it is constant

T F If f is holomorphic at z_0 and g has a pole at z_0, then $f + g$ has a pole at z_0

T F The function $1/\sin^2 z$ has a holomorphic anti-derivative in $\mathbb{C} \setminus \pi\mathbb{Z}$

T F If f_1 and f_2 both have convergent Taylor expansion around z_0 with radii of convergence R_1 and R_2, then the radius of convergence of the Taylor series for $f_1 + f_2$ is $\max(R_1, R_2)$.

T F Every holomorphic function defined on the outside of the unit disk has an anti-derivative there

T F If f has a pole at z_0 and g has an essential isolated singularity there, then $f \cdot g$ has an essential singularity at z_0

T F Every holomorphic function has a convergent Taylor expansion at each point in its domain

Show that, for $a > 0$,

$$\int_0^\infty \frac{x \sin ax}{x^4 + 4} dx = \frac{\pi}{4} e^{-a} \sin a.$$

Note: You must justify your calculations and estimates.

Question 4, 4 pts
Using residues, find the value of

$$\int_0^\pi \frac{\cos\theta \, d\theta}{2 + \cos\theta}.$$

Question 5, 3 pts
Prove the following *Mean value theorem*: If f is holomorphic on and within the circle C centered at z_0, the value of f at z_0 is the average of its values on C. Proceed as follows:
(a) By expressing the average as an integral, rephrase the statement as a formula:

$$f(z_0) = \int_0^{2\pi} ???$$

(b) Deduce the formula in (a) from Cauchy's formula.

Question 6, 2 pts, optional extra credit
Find a system of branch cuts that allows a single-valued choice of branch for the function $\sqrt[3]{\sin z \cos^2 z}$ and which does *not* involve cuts going out to ∞.