Math185 – Midterm 1 solutions

Question 1, 6 pts
(a) $i = \exp\frac{\pi i}{2}$, so the values of $i^{2/3}$ are $\exp\frac{\pi i}{3}, \exp\left(\frac{\pi i}{3} + \frac{2\pi i}{3}\right)$ and $\exp\left(\frac{\pi i}{3} - \frac{2\pi i}{3}\right)$; specifically they are the cube roots of (-1),

$$\frac{1}{2} + i\frac{\sqrt{3}}{2}, -1, \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

(b) $\log i = \frac{\pi i}{2} + 2n\pi i$, $n \in \mathbf{Z}$, so $i \log i = -\frac{\pi}{2} - 2n\pi, n \in \mathbf{Z}$ and so $\sin(i \cdot \log i) = -1$.
(c) $\exp\left(\frac{1}{2}\log i\right) = \exp\left(\frac{\pi i}{4} + n\pi i\right) = \pm(1+i)/\sqrt{2}$, the two square roots of i.

Question 2, 6 pts
f is complex-differentiable at z_0 iff

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists (and is finite), in which case the value of the limit is the complex derivative $f'(z_0)$.
If $f(z) = \bar{z}$, then $\frac{\bar{z} - \bar{z_0}}{z - z_0}$ has modulus 1 for any $z \neq z_0$; but the argument is twice the argument of $\bar{z} - \bar{z_0}$, so there is no limit as $z \to z_0$. (For instance, different directions of approach would give different limits.)
For the square root, we start by noting the identity $z - z_0 = (\sqrt{z} - \sqrt{z_0})(\sqrt{z} + \sqrt{z_0})$, valid for any choices of $\sqrt{z}, \sqrt{z_0}$. Therefore,

$$\frac{\sqrt{z} - \sqrt{z_0}}{z - z_0} = \frac{1}{\sqrt{z} + \sqrt{z_0}} \to \frac{1}{2\sqrt{z_0}}, \text{ as } z \to z_0.$$

Question 3, 7 pts
(a) Absolute and locally uniform, as it is dominated by $|z|^2 \cdot \sum 1/n^2$: so we can get a uniform bound on the error, as long as $|z|$ is bounded. It is not uniform in z, however, because for any fixed n we can get the nth term $z^2/(n^2 + |z|^2)$ to be close to 1 by making z large: so choosing $\varepsilon = \frac{1}{2}$ say, there is no n for which the errors above n are less than ε for all values of z.
(b) The numerator has exponential behavior in n: $(\exp z)^n$. So if $|\exp z| \leq 1$, the series is dominated by $\sum 1/n^2$ and converges absolutely and uniformly. For $|\exp z| > 1$, the general term increases without bound so the series diverges. The condition $|\exp z| \leq 1$ is equivalent to $\mathrm{Re}(z) \leq 0$.
(c) When z is real, $|\sin(nz)| \leq 1$ so the series converges absolutely and uniformly on the real axis. Off the axis, however, $\sin nz = \sin nx \cdot \cosh ny + i \cos nx \cdot \sinh ny$, so

$$\begin{aligned} |\sin(nz)|^2 &= \sin^2 nx \cdot \cosh^2 ny + \cos^2 nx \cdot \sinh^2 ny = \\ &= \sin^2 nx + \sin^2 nx \cdot \sinh^2 ny + \cos^2 nx \cdot \sinh^2 ny = \sin^2 nx + \sinh^2 ny \geq \sinh^2 ny \end{aligned}$$

and the latter grows exponentially with n as soon as $y \neq 0$.

Question 4, 6pts
The answer is always $2i \times$ (area enclosed by γ).