Math185 – Midterm 1

4 questions, 25 points, 75 minutes. Closed book, no notes. Justify your answers.

Question 1, 6 pts

Work out the possible complex values of the expressions below as explicitly as you can:

$$(a)\ i^{2/3} \quad (b)\ \sin(i \cdot \log i) \quad (c)\ \exp\left(\frac{1}{2}\log i\right)$$

Question 2, 6 pts

Write down what it means for a function $f : U \mapsto \mathbb{C}$, defined on an open subset $U \subset \mathbb{C}$, to be complex-differentiable at a point $z_0 \in U$, and define its complex derivative $f'(z_0)$.

Using the definition, show that the function $g(z) = \bar{z}$ is *not* complex-differentiable anywhere in $\mathbb{C}$.

Also from the definition, show that the function $\sqrt{z}$, defined for $z \in \mathbf{C} \setminus \mathbf{R}_{\leq 0}$ as the square root with positive real part, is complex-differentiable everywhere in its domain, with derivative $\frac{1}{2\sqrt{z}}$.

Question 3, 7 pts

Discuss the convergence of the series (absolute, uniform, or locally uniform)

$$(a)\ \sum_{n=1}^{\infty} \frac{z^2}{n^2 + |z|^2} \quad (b)\ \sum_{n=1}^{\infty} \frac{e^{nz}}{n^2 + 1} \quad (c)\ \sum_{n=1}^{\infty} \frac{\sin nz}{n^2 + 2}$$

Note: You may assume the convergence of the series $\sum 1/n^2$. Careful with the third one.

Question 4, 6pts

Find the complex integrals $\int_\gamma f(z)dz$ of the function $f(z) = \bar{z}$ along the following closed paths γ:

1. The circle of radius R centered at the origin, oriented counter-clockwise
2. The rectangle with vertices at $0, a, a + bi, bi$ $(a, b \in \mathbf{R}_+)$, in this order

You should now be able to guess the value of the integral along a closed curve bounding a more general region R (if not, then you made a mistake).