Math185 – Homework 9

Due on **MONDAY**, April 2

Question 1 (Schaum, 7.55) Find the residue of $e^{iz}/(z^2+1)^5$ at $z=i$ and use it to evaluate $\displaystyle\int_0^\infty \frac{\cos x dx}{(x^2+1)^5}$.

Question 2 (Schaum, 7.60) Prove that $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$.
Hint: Use a trigonometric identity to relate this to $(e^{2ix}-1)/x^2$.

Question 3 (Schaum, 7.90) Using the contour integral on a rectangle with vertices $-R, R, R+\pi i, -R+\pi i$, prove that

$$\int_0^\infty \frac{\cos px}{\cosh x}dx = \frac{\pi}{2\cosh(p\pi/2)}.$$

Question 4 (Schaum 7.100) Using a rectangular contour of height i and either the "notching" argument or half-residues, check that

$$\int_0^\infty \frac{x\,dx}{\sinh(\pi x)} = \frac{1}{4}$$

Question 5 Show using residues that

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}.$$

Question 6 Show by contour integration that for real $-1<\alpha<3$,

$$\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}dx = \frac{(1-\alpha)\pi}{4\cos(\alpha\pi/2)}.$$

explaining the appropriate interpretation of the right-hand side when $\alpha = 1$.
Hint: Use a semi-circular or a keyhole contour.

Question 7 By using a keyhole contour and the logarithm as an auxiliary function, determine

$$\int_0^\infty \frac{xdx}{x^3+x^2+x+1}$$

Question 8 (Sarason) Show that for $0<|a|<1$ and a positive integer n, the equation $(z-1)^n e^z = a$ has exactly n distinct solutions in the half-plane $\mathrm{Re}(z)>0$. (Use Rouché's theorem.)

Question 9*, (Schaum, 7.104) Verify the following formula for the Cauchy principal value integral, for $0<p<1$:

$$\mathrm{PV}\int_0^\infty \frac{x^{-p}dx}{x-1} = \pi\cot(p\pi)$$

(Use a keyhole contour.)

Question 10* (Schaum, 7.95) Using the contour bounded by an (infinitely) large circle plus a cut along the interval $[0,1]$, together with two vanishingly small circles around 0 and 1 (Schaum, Fig. 7.18), prove that

$$\int_0^1 \frac{dx}{\sqrt[3]{x^2-x^3}} = \frac{2\pi}{\sqrt{3}}$$