Math185 – Homework 8

Due in class on Wednesday, March 21

Question 1 Schaum, 5.35

Question 2 Schaum, 5.39

Question 3 Schaum, 5.88

Question 4 Prove that if a holomorphic function f has an isolated singularity at 0, then the principal part of its Laurent expansion converges everywhere on $\mathbb{C} \setminus \{0\}$.

Question 5 Give a formula for the residue at 0 of the function $\sin(z + z^{-1})$.

Question 6 Find the Laurent series of the function $f(z) = (z^2 - 1)\sin(1/z^2)$ in the domain $0 < |z| < \infty$.

Question 7 By choosing two different annuli, both centered at 0, in which the function below is holomorphic, find two different Laurent expansions for it in powers of z. Describe their regions of convergence.

$$f(z) = \frac{1}{z^2(1-z)}.$$

Question 8 Schaum, 6.92

Question 9 Find the residues of the following functions at each of their isolated singularities:

$$(a)\frac{z^p}{1-z^q} \quad (p, q \in \mathbb{Z}_{>0}) \qquad (b)\frac{z^5}{(z^2-1)^2} \qquad (c)\frac{\cos z}{1+z+z^2}$$

Question 10 Show that if $f(z)$ is a holomorphic function on $\mathbb{C}$, except possibly for finitely many poles, and $|f(z)| < (\text{const.})/|z|^2$ for large z, then

$$\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{\text{poles of } f(z)} \operatorname{Res}\left[f(z)\cot(\pi z)\right],$$

where it is understood that if f has a pole at some integer n, then the value $f(n)$ is excluded from the sum on the left (it contributes to the right instead).
Applying this to the function $f(z) = 1/(z - \pi w)^2$, for a fixed complex number w, deduce a formula of Euler's

$$\frac{1}{\sin^2 w} = \sum_{n\in\mathbb{Z}} \frac{1}{(w - n\pi)^2}.$$

Hint: Use the residue formula for $f(z)\cot(\pi z)$ for the square centered at 0, with sides parallel to the x, y-axes and crossing these at $\pm(N + 1/2)$ and $\pm(N + 1/2)i$, respectively. Estimate the contour integral and show that it vanishes as $N \to \infty$. If you are stuck, see Needham, 9.III.5 for help.

Question 11* Starting from the series expansion of Q8,

$$\frac{1}{\sin^2 z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n\pi)^2}, \tag{*}$$

integrate both sides twice and then exponentiate to derive another formula of Euler's,

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right).$$

Caution: In the product, the terms for n and $-n$ have been grouped together in order to get convergence. You need to do this with the series at the first integration, otherwise the series of integrals diverges.