Math185 – Homework 7

Due in class on Wednesday, March 14

Question 1
Use the maximum principle to prove the Fundamental Theorem of Algebra, as follows. Assume that the polynomial $p(z)$ does not vanish anywhere in $\mathbf{C}$. Show then that the function $1/p(z)$ must achieve a local maximum of its modulus at some complex value of z.
Hint: Recall that a continuous real function on a closed and bounded set achieves its maximum. Explain why you can resstrict to some such subset of $\mathbf{C}$ in seeking the maximum.

Question 2
Let $E \subset \mathbb{R}^n$ be an open set and $p \in E$ a point. Prove that that E is connected if and only if it is path-conected, as follows: show that the sets C_p and N_p of points which can, respectively cannot be connected to p by a continuous path are both open.
Repeat this for polygonal paths and conclude that the notions of path-connectivity and polygonal-path-connectivity agree for open sets.

Question 3
In contrast with the previous question, show that the closed subset of those $(x, y) \in \mathbb{R}^2$ defined by $y = \sin(1/x)$ for $x \neq 0$ and $y \in [-1, 1]$ for $x = 0$ is connected, but not path-connected.

Question 4: Schaum, 5.48

Question 5: Schaum, 5.49

Question 6*: Schaum, 5.55

Question 7: Schaum, 5.85

Question 8*
Let γ be a simple (free of self-intersections) curve of class C^1 in $\mathbb{C}$, not necessarily closed. Let $\varphi : \gamma \to \mathbb{C}$ be a continuous function. Show that the *Cauchy integral*

$$f(z) := \frac{1}{2\pi i} \int_\gamma \frac{\varphi(\zeta) d\zeta}{\zeta - z}$$

is a holomorphic function on $\mathbb{C} \setminus \gamma$, and that $\lim_{z \to \infty} f(z) = 0$.

Hint: Expand the integrand in a Taylor series centeredd at any z_0 not on γ.
Alternative: use *Weierstraß convergence* on Riemann sum approximations of the integral (Sarason, VII.15; also use the fact that continuous functions on a compact interval are uniformly continuous).
Remark: When φ is continuously differentiable, the function f has the remarkable property that its limiting values from opposite sides at any point $p \in \gamma$ which is *not* an endpoint differ by $\varphi(p)$; that is, f has a jump discontinuity across γ, with jump φ. Try to prove this!
The singularity at $z = a$ of the function $(z - a)^{-1}$ is called a *pole*. This example can be summarised as "an integral of poles is a cut"

Question 9
In Question 8, determine $f(z)$ in the following two situations:

1. γ is closed, and φ is the restriction of a function which is holomorphic on γ and its interior
2. γ is not necessarily closed, but φ is the constant function 1.

Note: The answer for (2) is tricky to state for general γ. Try first the case of a line segment.

Question 10

Let E be a connected open region in $\mathbb{C}$ which is symmetric under reflection about the real axis, and let $f : E \to \mathbb{C}$ be a holomorphic function which is real-valued on $E \cap \mathbb{R}$. Prove that $f(\bar{z}) = \overline{f(z)}, \forall z \in E$. Do so in two steps:

1. Prove that the function $g(z) := \overline{f(\bar{z})}$ is holomorphic in E (mind the double bar)
2. Compare the functions $f(z), g(z)$ on $E \cap \mathbb{R}$.

Question 11*

Prove the *Schwarz reflection principle*: let E be as in Q10, and now assume that f only defined and holomorphic on the upper half of E, but extends continuously, with real values, to $E \cap \mathbb{R}$. Then, f extends uniquely to a holomorphic function on all of E.

Note: Q10 gives you the only possible candidate for this extension. Now you must check that the so patched f is holomorphic. The only problem is on the real axis. You may assume that f is continuously real-differentiable there. (You can avoid that assumption if you use Morera's theorem – Sarason VII.10.)