Math185 – Homework 6, the Cauchy homework

Due in class on Wwednesday, March 7

Question 1

Let C be a simple, closed, piecewise C^1 curve in the complex plane, enclosing a region D. Use the complex version of Green's theorem to show that

$$\oint_C \bar{z} dz = 2i \cdot \text{Area}(D)$$

assuming that C has been oriented counter-clockwise.
(You may assume that D is star-shaped, or what makes you comfortable in using Green's theorem.)

Question 2 (Sarason VI.12.1)

By integrating the function $\exp(-z^2)$ around the circular sector of radius R, centered at 0, and bounded by the rays $\arg z = 0$ and $\arg z = \pi/8$, and letting $R \to \infty$, show that

$$\int_0^\infty e^{-t^2} \cos t^2 dt = \frac{1}{4}\sqrt{\pi}\sqrt{1+\sqrt{2}}$$

Explain why the contribution of the circular arc vanishes as $R \to \infty$.
Note: For the value of $\int_0^\infty e^{-t^2} dt$, see the text just preceding the question.

Question 3 (Sarason VI.12.2)

By integrating the same function around the sector now with angle $\pi/4$, evaluate the Fresnel integrals

$$\int_0^\infty \cos t^2 dt, \qquad \int_0^\infty \sin t^2 dt.$$

This time, you need a careful argument for the vanishing of the contribution of the circular arc; this is related to the slow convergence of the real improper integral.

Question 4

Apply Cauchy's formula to a large ($R \to \infty$) half-disk in the upper half plane and the function $\exp(iz)/(z^4+4)$ to find the value of

$$\int_0^\infty \frac{\cos(x)}{x^4+4} dx$$

Question 5, (Schaum, 5.32)

Determine $\oint_C \frac{e^{3z}}{z-\pi i} dz$ if C is: (a) the circle $|z-1| = 4$; (b) the ellipse $|z-2|+|z+2| = 6$.

Question 6 (Schaum, 5.33)

Determine $\oint_C \frac{\cos \pi z}{z^2-1} dz$ around the rectangles with vertices at: (a) $2 \pm i, -2 \pm i$; (b) $\pm i, 2 \pm i$

Question 7

Apply Cauchy's formula to the function $ze^{iz}/(z^4+4)$ on a large ($R \to \infty$) upper half-disk to show that

$$\int_0^\infty \frac{x \sin x}{x^4+4} dx = \frac{\pi}{4e} \sin 1$$

Question 8

Apply Cauchy's formula to a large ($R \to \infty$) first quadrant quarter-disk to show, for a fixed real number $a > 0$,

$$\int_0^\infty \frac{dx}{x^4 + a^4} = \frac{\pi}{2\sqrt{2}a^3}, \quad \text{and} \quad \int_0^\infty \frac{x dx}{x^4 + a^4} = \frac{\pi}{4a^2}$$

Question 9

For exponents $\alpha \in \mathbb{R}$ and $z = re^{i\theta} \neq 0$, $-\pi < \theta < \pi$, define $z^\alpha := r^\alpha(\cos(\alpha\theta) + i\sin(\alpha\theta))$.
Check directly that the map $z \to z^\alpha$ is holomorphic, and that $\frac{d}{dz} z^\alpha = \alpha z^{\alpha-1}$.
Check that, compared to our earlier (multi-valued) definition $z^\alpha := \exp(\alpha \log z)$, this corresponds to the choice of principal branch Log, and that for a rational number $\alpha = \frac{m}{n}$, z^α thus defined is an nth root of z^m.
From the consequences of Cauchy's theorem (Sarason VII.8) deduce the binomial formula, for exponents $\alpha \in \mathbb{C}$ and $z \in \mathbb{C}$, $|z| < 1$:

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2} z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{2!} z^3 + \dots$$

Also verify the radius of convergence directly (using the ratio test or Hadamard's formula).

Question 10*, optional

Let z_0 be a fixed complex number and let the complex function f be defined and continuous in the disk $|z - z_0| < R$, and let C_r be the circle of radius $r < R$ centered at z_0. Show that

$$\lim_{r \to 0} \oint_{C_r} \frac{f(z)}{z - z_0} dz = 2\pi i \cdot f(z_0).$$

Note: We do not assume that f is holomorphic, or even real-differentiable.