Math185 – Homework 5

For self-assessment and midterm preparation: Not due in class.
You may find some questions more challenging. This is good for the soul.

Question 1
Show that a convergent power series $\sum_{n=0}^{\infty} a_n z^n$ sums to the zero function if and only if $a_n = 0$ for all n. (*Hint:* Keep differentiating and evaluate at $z = 0$.)

Question 2 (Sarason, V.16.3)
Fix a positive integer k. Show that the following power series has radius of convergence ∞:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{k+2n}}{n!(n+k)!}.$$

By differentiating term-by-term, prove that its sum $J_k(z)$ satisfies *Bessel's differential equation*

$$z^2 J_k''(z) + z J_k'(z) + (z^2 - k^2) J_k(z) = 0.$$

Question 2
Find the series expansion of $f(z) = 1/(1 - z + z^2)$ by two different methods:

- By partial fractions expansion, and using the geometric series
- By setting up a recursion for the coefficients, using the Cauchy product (Sarason, V.18)

Question 3
By choosing convenient parametrizations, evaluate the following integrals:

1. $\oint z^{-1} dz$, around the square with vertices at $\pm 1 \pm i$
2. $\oint z^m dz$ around the unit circle, m an integer. (You should get 0, if $m \neq -1$.)

Question 4
(Schaum, 4.36) Evaluate $\int (z^2 + 3z) dz$ along
(a) the circular arc $|z| = 2$, from 2 to $2i$; (b) the straight line from 2 to $2i$; (c) the straight lines from 2 to $2 + 2i$ and then from $2 + 2i$ to $2i$.

Question 5
Derive the *Wallis formula*

$$\int_0^{2\pi} \cos^{2n} \theta d\theta = 2\pi \frac{(2n)!}{2^{2n}(n!)^2}$$

by integrating $\frac{1}{z}\left(z + \frac{1}{z}\right)^{2n}$ around the unit circle, parametrised in the usual way, and using the binomial formula and invoking Q.3, part 2.

Question 6*, optional
Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series of complex numbers. Let $\sum_{n=0}^{\infty} b_n$ be the same series after a reordering of the terms. (This means that every term in appears the exact same number of times in either series, in case there are repetitions.) Show that the two series have the same sum.