Math185 – Homework 4

Due in class on Wednesday, February 14

Question 1

Let u be a real-valued C^2 function on an open set $D \subset \mathbb{R}^2$.
Show that the function $g := \partial u/\partial x - i \cdot \partial u/\partial y$ is a holomorphic function of $z = x + iy$ in D if and only if u is harmonic.
Further, if u is the real part of the holomorphic function f, show that $g = df/dz$.
Remark: In conclusion, the *harmonic conjugate* v of u is the imaginary part of the complex *antiderivative* of g. Soon, we will learn how to find the complex anti-derivative using line integrals.

Question 2

Find the harmonic conjugates of the following functions:

$$(a) \quad x^3 - 3xy^2; \qquad (b) \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Question 3

Investigate the (a) absolute and (b) uniform convergence of the series

$$\frac{z}{3} + \frac{z(3-z)}{3^2} + \frac{z(3-z)^2}{3^3} + \frac{z(3-z)^3}{3^4} + \dots$$

Question 4

For which real values of x do the following power series converge? (You also need to check the borderline cases which the ratio test fails to settle.)

$$(a) \quad \sum_{n \geq 0} \frac{x^n}{n^2}; \qquad (b) \quad \sum_{n \geq 0} \frac{x^n}{2^n}; \qquad (c) \quad \sum_{n \geq 0} \frac{x^n}{\sqrt{n!}}; \qquad (d) \quad \sum_{n \geq 0} \sqrt{n!} \cdot x^n.$$

Question 5

Find the radii of convergence of $\sum n^n z^n$ and of $\sum n^2 z^n$. Sum the second series in closed form.

Question 6

Show that if $a(z) = \sum a_n z^n$ converges for small z and $a_n \neq 0$ for some $n > 0$, then for all sufficiently small $z \neq 0$ we have $a(z) \neq a_0$. (In other words, the solution $z = 0$ to $a(z) = a_0$ is *isolated*.)
Hint: Write $a(z) = a_0 + z^k(a_k + \sum_{n>k} a_n z^{n-k})$ and explain, exploiting continuity of the sum, why the second term is not zero for small, non-zero z.

Question 7

If the power series $a(z)$ and $b(z)$ converge for $|z| < R$, we have seen that their product $a(z)b(z)$ also converges for $|z| < R$. Find an example in which the radius of convergence for $a(z)b(z)$ is *greater* that that of either $a(z)$ or $b(z)$.
Hint: Try ratios of linear functions.