Math185 – Homework 3

Due in class on Wednesday, February 7

Recall that a function is said to be *of class* C^1 if it is differentiable with continuous partial derivatives, of class C^2 if it is twice differentiable with continuous partials, etc.

Question 1
Show that, in polar coordinates (r, θ), the Cauchy-Riemann equations for the differentiable function $(r, \theta) \mapsto u + iv$ read as follows, when $r > 0$

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Question 2
Let f be a holomorphic function on an open set $D \subset \mathbb{C}$. Assuming that f is of class C^2, show that the complex derivative df/dz is also holomorphic in D.
Remark: The C^2 assumption turns out to be unnecessary, but we cannot prove this yet.

Question 3* (Needham, IX.5,6)
Consider the map $f(x + iy) = (x^2 + y^2) + i(y/x)$. Find and sketch the curves that are mapped by f into (a) horizontal lines and (b) vertical lines. Notice from your answer that f appears to be angle-preserving. Show that it is not in two ways: by explicitly finding some curves whose angle of intersection is not preserved and by checking that the Cauchy-Riemann equations fail.
Show furthermore that no choice of v can make $f(x + iy) = (x^2 + y^2) + iv$ holomorphic.

Question 4
Verify, by the method of your choice, that the function $(x+iy) \mapsto e^x \cdot (\cos y + i \sin y)$ is holomorphic in the entire complex plane.

Question 5
Represent $\exp(\frac{\pi i}{4}), \exp(\frac{\pi i}{2})$ and their sum in the complex plane. By expressing each of them as $x + iy$, deduce that $\tan \frac{3\pi}{8} = 1 + \sqrt{2}$.
By considering $(2 + i)(3 + i)$, show that $\frac{\pi}{4} = \tan^{-1}(1/2) + \tan^{-1}(1/3)$.

Question 6
Which of the following functions are holomorphic functions of $z = x + iy = r(\cos \theta + i \sin \theta)$?

$$e^{-y}(\cos x + i \sin x); \quad \cos x - i \sin y; \quad r^3 + 3i\theta; \quad re^{r\cos\theta}(\cos(\theta + r\sin\theta) + i\sin(\theta + r\sin\theta)).$$

Question 7 (Sarason, IV.8.2)
Establish the identities ($x = \mathrm{Re}(z), y = \mathrm{Im}(z)$)

$$\begin{aligned}
\cos z &= \cos x \cosh y - i \sin x \sinh y, \\
\sin z &= \sin x \cosh y + i \cos x \sinh y, \\
|\cos z|^2 &= \cos^2 x + \sinh^2 y, \\
|\sin z|^2 &= \sin^2 x + \sinh^2 y.
\end{aligned}$$

Question 8
Show that the image under $z \mapsto \cos z$ of a horizontal line is an ellipse, and the image of a vertical line is a hyperbola. There are some exceptional cases: discuss those.