Math185 – Homework 2

Due in class, Wednesday January 31

Question 1

Let $a = \cos\theta + i\sin\theta$ and let b be an arbitrary complex number. If $a \neq 1$, show that the map $z \mapsto az + b$ represents geometrically a rotation of the plane by angle θ with center $z_0 = b/(1-a)$. What happens if $a = 1$?

Question 2

Find all the cube roots of i.
Find $(1+i)^{100}$.

Question 3

Factor the polynomial $x^4 + 1$ into two quadratic polynomials with *real* coefficients.
Hint: Split it first into four complex linear factors, and then combine those in suitable pairs.

Question 4

Prove the formula asserted in lecture: the area of the triangle T in $\mathbb{R}^2 \cong \mathbb{C}$, with vertices defined by the complex numbers a, b, c (counterclockwise) is $\frac{1}{2}\mathrm{Im}(\bar{a}b + \bar{b}c + \bar{c}a)$.
Suggestion: Assume first that one of the vertices is the origin. Then, check that the formula is translation-invariant: it is not changed when all of a, b, c are shifted by the same complex number.

Question 5

Regard the map $z \mapsto f(z) = 1/z$ as a map from $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{R}^2$. Show that it is real-differentiable everywhere and compute its Jacobian matrix. Verify the Cauchy-Riemann equations.
Using complex numbers, show that f is differentiable in the complex sense and determine its derivative. Verify that this agrees with the real calculation.

Question 6

At what points in $\mathbb{C}$ are the following functions complex-differentiable? ($r = |z|$ in part d.)
(a) $f(z) = \mathrm{Re}(z)$ (b) $f(z) = \bar{z}$ (c) $f(z) = \bar{z}(z^2 - 1)$ (d) $f(z) = \bar{z}/r^2$, $f(0) = 0$.

Question 7

Let the function f be holomorphic in an open disk $D \subset \mathbb{C}$. Show that each of the following conditions forces f to be constant.
(a) $f' \equiv 0$ in D (b) f is real-valued in D (c) $|f|$ is constant in D
(d) $\arg f$ is constant in D (e) $\overline{f(z)}$ is also holomorphic.
Hint: For the last four cases, use the Cauchy-Riemann equations.

Question 8, optional

Show that a linear map from $\mathbb{R}^n$ to itself which preserves orthogonality of vectors must be the composition of an orthogonal map with a scaling. In particular, it preserve all angles (in absolute value).