Math185 – Homework 11

Due on Monday, April 30

Question 1, Schaum 8.30

Sketch the region of the w-plane into which the interior of the triangle with vertices at $z = i, 1-i, 1+i$ is taped by the transformations (a) $w = z^2$, (b) $w = iz^2 + (2-i)z$, (c) $w = z + 1/z$.

Question 2, Schaum 8.33a

Show that the transformation $z \mapsto w = ze^{-\alpha} + z^{-1}e^{\alpha}$, for a fixed real α, maps the interior of the unit disk to the exterior of an ellipse. (0 maps to ∞.)

Question 3

Find, by means of the Poisson formula, a harmonic function in the unit disk whose restriction to the unit circle is $\cos(2\theta)$. Is there an easier way to see the answer?

Question 4 (Poisson kernel on the upper half plane)

Let $f(t)$ be a bounded, continuous real-valued function on $\mathbb{R}$. Prove that the function

$$\Phi(x, y) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y \cdot f(t)}{(x-t)^2 + y^2} dt$$

"has f as its boundary value on the real axis": that is, $\lim_{y \to 0} \Phi(x, y) = f(x), \forall x \in \mathbb{R}$.

Question 5, Schaum 9.38

Find a harmonic function on the upper half-plane whose values on the real axis are -1, for $x < -1$, 0 for $-1 < x < 1$ and 1 for $x > 1$.
Hint: Use the Poisson kernel in Q.4

Question 6, Schaum 9.35

Prove that a harmonic function in a connected, open region of the plane that depends only on the distance r to some fixed point has the form $A \log r + B$, for some constants A, B.

Question 7

Check by computation that the map $z \mapsto w = \sqrt{z^2 - 1}$ maps the (open) upper half plane conformally onto the upper half plane minus the line segment from 0 to i. Explain which points on the real axis go where (some will map to the vertical segment).