Math185 – Homework 10

Due on Wednesday, April 18

Question 1

Check that a Möbius transformation $z \mapsto \phi(z) = \frac{uz+v}{-\bar{v}z+\bar{u}}$ with $|u|^2 + |v|^2 = 1$, correspond to an *isometry* (distance-preserving map) of the sphere under stereographic projection. To do so, choose points P, Q corresponding to $z, w = \phi(z)$ on the plane, and show that the scaling factor for lengths of tangent vectors to the sphere at P and Q under the map is 1. For this, Recall from class that the scaling factors for stereographic projection from P to z and from Q to w are $\frac{1}{2}(|z|^2+1)$, $\frac{1}{2}(|w|^2+1)$, respectively, and recall also that the derivative $\phi'(z)$ is $1/(-\bar{v}z + \bar{u})^2$, and proceed as follows:

1. Explain why the scaling factor under ϕ from z to w is $|\phi'(z)|$;
2. Combine the scaling factors from P to z, z to w, w to Q and show that you get 1.

Question 2

Check that a Möbius transformation can be written as a composition of the following three special types: translation $z \mapsto z + c$, complex scaling $z \mapsto k \cdot z$ ($k \in \mathbb{C}$), and complex inversion $z \mapsto 1/z$.

Question 3

Verify that any Möbius transformation preserves the cross-ratio of four points:

$$(z_1, z_2; z_3, z_4) := \frac{z_4 - z_2}{z_4 - z_1} \bigg/ \frac{z_3 - z_2}{z_3 - z_1}$$

Hint: By Q.3, it suffices to check this for the three transformations listed.

Question 4

Find **all** Möbius transformations which take the imaginary line (with ∞) to the unit circle, and the point $+1$ to the origin. (*Note:* You should find a one-parameter family.)

Question 5

Project $\mathbb{C}$ stereographically to the Riemann sphere. Then, project the sphere back to $\mathbb{C}$, but from the *South pole* instead of the North pole. The result is a map from $\mathbb{C} \setminus \{0\}$ to itself. Describe this map (a) by a formula (b) geometrically. What does the antipodal map on the sphere correspond to, in the complex plane? (*Note:* The two answers are closely related.)

Question 6*

Show that any pair of *non-intersecting* circles can be mapped to a pair of concentric circles by some Möbius transformation.

Hint: Show that you can find a circle C which is orthogonal to both of them. (It's easy if you Möbius map one of the circles to a line.) Then, show that you can find a Möbius map sending C to the imaginary axis and the line joining the centers of the two circles to the real axis. Conclude that both original circles are now centered at 0.

Question 7

In geometry, *inversion* with respect to a circle with center C and radius r is the transformation on $\mathbb{R}^2 \setminus \{C\}$ defined as follows. A point P gets sent to the unique point P' lying on the ray joining C to P and such that the product of the distances from C to P and P' equals r^2.

Show that inversion sends clircles to clircles. Specifically, circles not passing through C map to circles not passing through C, circles through C map to lines not passing through C, lines through C map to lines through C and lines not containing C map to circles through C.

Hint: Reduce to the case of the unit circle, and relate inversion to a Möbius transformation.

Question 8, Sarrason III.7

Let $f(z) = \left(\frac{z+1}{z-1}\right)^2$, $f(1) = \infty$, $f(\infty) = 1$. Describe the images of the following sets:
(a) The extended real axis (b) the extended imaginary axis (c) The right half-plane $\mathrm{Re}z > 0$.
Remark: 'Extended axis' means including the point ∞.