Math 185, S18 — Homework 1

Due on 24 January

Question 1

Show, by real calculus methods, that the cubic equation (with real constants p, q)

$$x^3 - 3px - 2q = 0$$

has one real solutions when $q^2 > p^3$ and three real solutions when $q^2 < p^3$.

Question 2

Let $\omega = \cos \frac{2\pi}{3} + \mathrm{i} \sin \frac{2\pi}{3}$ (it is a cube root of 1).

- If $\Delta := q^2 - p^3 > 0$, let $r_\pm$ be the real cube roots of the two numbers $q \pm \sqrt{\Delta}$. Show that $r_+ + r_-$ is a real solution of the cubic equation in Q1, while $\omega r_+ + \omega^2 r_-$ and $\omega^2 r_+ + \omega r_-$ form a complex-conjgate pair of solutions.
- If $\Delta < 0$, let r_+ be a complex cube root of $q + i\sqrt{-\Delta}$ and r_- the complex conjugate of r_+. Show that the three solutions of the cubic are real, namely $r_+ + r_-$, $\omega r_+ + \omega^2 r_-$ and $\omega^2 r_+ + \omega r_-$.

Question 3

(a) Show that if f is a polynomial with *real* coefficients and $f(\alpha) = 0$ for some $\alpha \in \mathbb{C}$, then $f(\bar{\alpha}) = 0$ as well. *Hint:* Conjugate the entire equation.

(b*) The fundamental theorem of algebra implies that a polynomial with complex coefficients,

$$p(z) = z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0,$$

splits into a product of linear factors,

$$p(z) = (z - \rho_1) \cdot \cdots \cdot (z - \rho_n),$$

with complex numbers ρ_k. Use this to show that, if all coefficients c_k are real, then $p(z)$ splits as a product of linear and quadratic factors with *real* coefficients.

Hint: Combine the non-real roots into complex-conjugate pairs.

Question 4

By taking real and imaginary parts in the geometric sum formula

$$1 + z + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1},$$

and using de Moivre's formulas, show that

$$\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\sin(n + 1/2)\theta}{2\sin(\theta/2)},$$

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \frac{\cos(\theta/2) - \cos(n + 1/2)\theta}{2\sin(\theta/2)}$$

Hint: Rewrite $(z^{n+1} - 1)/(z - 1)$ as $(z^{n+1/2} - z^{-1/2})/(z^{1/2} - z^{-1/2})$; this makes it easy to take real and imaginary parts!

Question 5
Let $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$ for some natural number n. From the geometric sum formula (Q3), show that $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$.

Question 6
With ω as in Q5, calculate $(1+\omega)^k$ using de Moivre. (Draw a picture for $1+\omega$ to work out argument and modulus.)