Math185 – Midterm 1

Solve any FIVE of the problems for 25 points.
75 minutes. Closed book, no notes.

Question 1, 5 pts
Find, in the form $x + iy$, all square roots of $1 + i$ and of $3 + 4i$, and represent them on the plane.
Comment: Sometimes the polar form is best for taking square roots, but sometimes not.

Question 2, 5 pts
Write down the definition of a harmonic function, and explain briefly why the real and imaginary parts of a (twice differentiable) holomorphic function are harmonic.
Show that the following function is harmonic and find a harmonic conjugate:

$$\frac{y}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$$

Hint: It may help to express in terms of $z, \bar{z}$.

Question 3, 5 pts
Determine the radius of convergence of the following two series. For series (b), also explain what happens on the boundary of the disk of convergence.

$$(a) \quad \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} z^n \qquad (b) \quad \sum_{n=1}^{\infty} \frac{z^{3n}}{3^n \cdot n^3}$$

Question 4, 5 pts
Find all the values of z in the complex plane where the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{(z^2 + 1)^n}$$

Sum the series. If you feel artistically inclined, sketch the region of convergence.

Question 5, 5pts
Compute the integral $\int z^{-1} dz$ along the following paths joining the points $1 - i$ and $1 + i$:
(a) the straight line;
(b) the quarter-circle of radius $\sqrt{2}$;
(c) the 3/4 circle of radius $\sqrt{2}$, clockwise around 0.
Explain why your answers for (a) and (b) agree, but the one for (c) is different.

Question 6, 5pts
By using Cauchy's theorem, applied to a contour in the upper half-plane consisting of the interval $[-R, R]$ and a half-circle of radius R, and letting $R \to \infty$, verify that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + x + 1} = \frac{2\pi}{\sqrt{3}}.$$

Also explain how you would do this integral by real calculus methods.