Math 185 – Final Exam

Monday, May 8 2017, 11:30–2:30

Question 1, 10 pts
For the following functions, locate the singularities, if any, determine their type and find the residues if appropriate.

$$(a)\ \frac{1}{\tan z - 1} \qquad (b)\ \frac{e^z - 1}{e^{2z} - 1}$$

Note: Explain clearly how you located *all* the complex zeroes of the denominators.

Question 2, 10 pts

1. Using the Cauchy-Riemann equations, show that for a holomorphic function $f = u+iv$, we have $f'(z) = u_x - iu_y$. (As usual, $z = x + iy$.)

2. Find a holomorphic function $f(z)$ defined for $z \neq 0$ whose real part is $\dfrac{2xy}{(x^2+y^2)^2}$.
   Hint: If you cannot guess the answer, use Part 1.

Question 3, 10 pts
Using residues, or otherwise, prove that for real numbers a, b with $b > 0$,

$$\int_{-\infty}^{\infty} \frac{dt}{(t^2+1)((t-a)^2+b^2)} = \frac{\pi}{b} \cdot \frac{(b+1)}{(a^2+(b+1)^2)}$$

Question 4, 10 pts
Find a Möbius transformation which takes the unit circle centered at 0 to itself and the line $x = 5/3$ to the circle of radius $1/3$ centered at the origin.
Hint: It is possible to arrange that the real axis gets mapped to itself. Assuming this, determine where some important points have to go under the map.

Question 5, 8 pts
Using the Poisson kernel for the upper half-plane and the integral in Q3 above, find a bounded harmonic function $f(x, y)$ on the upper half-plane which restricts to $1/(x^2+1)$ on the real axis.

Question 6, 12 pts

1. When a, b are distinct points lying on on the unit circle, explain why the function $f_{a,b}(z) := \arg\left(\dfrac{z-a}{z-b}\right)$ can be defined as a *single-valued* and *harmonic* function on the closed unit disk, except for the points a, b.

2. Describe the values of the function $f_{a,b}(z)$ you just defined for z *on* the unit circle (but away from a and b).

3. Find a harmonic function on the unit disk which takes the following limiting values on the unit circle:

$$1,\ \text{when } 0 < \arg(z) < \frac{\pi}{2}$$
$$0,\ \text{when } \frac{\pi}{2} < \arg(z) < 2\pi$$