Math 185, Fall 2014: First exam, 9/18

25 points, 75 minutes

You must justify your answers.

Question 1 (3+3)
(a) Express z^7 in polar form, $r(\cos\theta + \mathrm{i}\sin\theta)$, where

$$z = \sqrt{\frac{\sqrt{2}+1}{2}} + \mathrm{i}\sqrt{\frac{\sqrt{2}-1}{2}}.$$

Hint: First, compute z^2. And don't make mistakes...
(b) Find all the sixth roots of 8i and represent them graphically in the plane.
Question 2 (4)
Prove that, for every integer $n > 1$,

$$1 + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cdots + \cos\frac{2(n-1)\pi}{n} = 0.$$

Hint: Consider the powers of $\omega = \cos\frac{2\pi}{n} + \mathrm{i}\sin\frac{2\pi}{n}$.
Question 3 (4+4)
(a) Let $f : D \to \mathbb{C}$ be a holomorphic function in the unit disk D and assume it has continuous second (real) partial derivatives. Prove that the real part of f is harmonic.
Remark: You may use the Cauchy-Riemann equations, if you state them and their relationship to the holomorphy condition on f.
(b) Show that the function $xy/(x^2+y^2)^2$ is harmonic in $\mathbb{R}^2$ away from $(0,0)$.
Question 4 (3+2+2)
(a) Find a Möbius transformation $M(z)$ that takes the points $-1, 0, 1$ to $0, 1, 4$ (as ordered).
(b) Find the fixed points of your transformation, the z with $M(z) = z$.
(c) What is the image under M of the circle of radius 1, centered at 1?
Caution: Check your work for (a), before proceeding with (b) and (c)!
Bonus Question
(a) You are given two distinct complex numbers β, δ. Assuming that $|x-\beta| = |x-\delta|$ for all real numbers x, prove that $\beta = \bar{\delta}$. *Hint:* Draw a picture.
(b) Show that any Möbius transformation which takes the real axis (with ∞) to the unit circle can be written in the form

$$M(z) = \alpha\frac{z-\beta}{z-\bar{\beta}}$$

for certain complex number α of modulus 1, and complex (and non-real) number β.
Hint: Specialize a general map to $z = \infty$; then use Part (a).