Math 185 – Final Exam

Friday, December 19 2014, 7–10pm

This is a closed book exam.

Please write clearly and explain your reasoning, unless instructed otherwise

6 Questions, 45 points, 180 minutes.

Name: ____

Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Tot.	

Question 1, 6 pts
True or false? Circle the correct answer; no reason necessary.

T F A polynomial in z and $\bar{z}$ represents a holomorphic function if and only if it is independent of $\bar{z}$ (it has an expression where the variable $\bar{z}$ does not appear).

T F If $\sum_n a_n z^n$ and $\sum_n b_n z^n$ converge for all $|z| < R$, then $\sum_n (a_n - b_n) z^n$ converges for all $|z| < R$.

T F The function $\cos(z^3 + 1/z^3)$ has a Laurent series centered at 0 which converges everywhere on $\mathbb{C} \setminus \{0\}$.

T F If, for an entire holomorphic function f, the derivative f' is bounded on $\mathbb{C}$, then f is linear or a constant function.

T F If the function f is defined and holomorphic near p and $f(p) = 0$, then for a sufficiently small circle C centered at p, $\dfrac{1}{2\pi i} \displaystyle\oint_C \dfrac{f'(z)dz}{f(z)}$ is a positive number.

T F If, for a holomorphic function f, the modulus $|f|$ has a local maximum at some point in the interior of its domain, then f is constant near that point.

Question 2, 9 pts

Describe the types of singularities of each of the following functions (at finite values of z) and determine the residues at all isolated singularities:

$$(a)\ \tan(z) \qquad (b)\ \frac{\sin(z+z^{-1})}{z+z^{-1}} \qquad (c)\ \frac{z}{(z^2-1)^2}$$

Hint for (b): No computation is needed to find the residue at the one nasty singularity.

Question 3, 7 pts

By integrating on the contour surrounding a circular sector of angle $2\pi/n$ in the upper half plane and letting $R \to \infty$, show that for any real $0 \leq \alpha < n-1$,

$$\int_0^\infty \frac{x^\alpha dx}{1+x^n} = \frac{\pi}{n\sin(\pi(\alpha+1)/n)}.$$

What, if anything, needs to change if $-1 < \alpha < 0$?

Question 4, 6 pts

Prove that
$$\int_0^\infty \frac{(\log x)^2 dx}{x^2+1} = \frac{\pi^3}{8}.$$
Hint: Integrate on an upper-half circle, with a slight notch to avoid 0. Justify the limits you need.

Question 5, 6 pts
Prove that the sum of all residues of the following function on $\mathbb{C}$ equals $\frac{3}{5}$:

$$\frac{3z^8 - 4z^7 + 5z^3 - 2z + 1}{5z^9 - 11z^6 + 12z^5 + z^3 - z}$$

Hint: Do NOT attempt to compute all the residues; instead, use the residue formula on a suitable contour.

Question 6, 10 pts

1. Check that the function $\arg(z)$ in the upper half-plane $\mathfrak{H}$ is harmonic (including the real line, but excluding the origin). Quickly describe the limiting behavior as you approach the discontinuity at 0.

2. By judicious use of functions $\arg(z-a)$, $a \in \mathbb{R}$, find a bounded, harmonic function on $\mathfrak{H}$ which, on the real axis, takes the limiting values 0 for $|x| > 1$ and 1 for $|x| < 1$. Briefly say how you would generalize your method to find a harmonic functions with prescribed, constant boundary values on (a finite collection of) real intervals.

3. Let $G := \{z \,|\, \mathrm{Im}(z) > 0 \text{ and } |z| > 1\}$ be the exterior of the unit half-disk in $\mathfrak{H}$. Find a conformal map identifying G with $\mathfrak{H}$, and use it to find a harmonic function in G with limiting values 1 on the half-circle and 0 on the real axis.