Math185 – Midterm 1

Thursday, October 5, 2006, 9:40-11:00
This is a closed book exam.
Please write clearly and explain your reasoning, unless instructed otherwise

Question 1, 4 pts
Are the following functions harmonic? Explain your answer.

$$(a)\; x^3 - 6xy^2; \quad (b)\; e^{2r\cos\theta} \cdot \cos(2r\sin\theta)$$

Question 2, 5 pts
True or false? Circle the correct answer; no reason necessary.

T F If $\operatorname{Im} a < 0$, then a has exactly one square root in the upper half-plane

T F If f and $\bar{f}$ are both holomorphic in a disk D, then f is constant

T F A conformal map which maps the unit disk onto itself must be a rotation

T F Under stereographic projection, every Möbius map corresponds to a rigid rotation of the sphere

T F The function $(x + iy) \mapsto x^2 + y^2$ is the real part of some holomorphic function

T F The product of two holomorphic functions is holomorphic

T F Any Möbius map which fixes $0, 1$ and ∞ must preserve distances

T F If the function u is harmonic, then so is u^2

T F Any Möbius map which takes a circle C to a circle C' must map the interior of C to the interior of C'

T F If $\sum a_n z^n$ converges for $z = z_1$ and if $|z_1| = |z_2|$, then it also converges for $z = z_2$

Question 3, 4pts
Let $n > 1$. By taking imaginary parts in the identity

$$1 + z + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z},$$

with a suitable complex number z, show that

$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} = \frac{\sin\frac{\pi}{n}}{1 + \cos\frac{\pi}{n}}.$$

Question 4, 6 pts
Let M be the Möbius transformation which takes the points $-1, 1, 2$ to the points $0, 1, \infty$. Find $M(0)$. What is M of a line? What happens to the upper half-plane?

Question 5, 6 pts
Find the radii of convergence of the following series:

$$(a) \sum 2^n z^n; \quad (b) \sum 2^n z^{2n}; \quad (c) \sum 2^n z^{n^2}.$$

For the series in (a), also discuss what happens on the boundary of the disk of convergence.

Question 6, Optional Extra Credit: 2.5 pts
Assume that the power series $\sum a_n z^n$ converges in some non-empty disk $|z| < R$. If the sum $a(z)$ is the zero function, show that all coefficients a_n must be zero.