# A NON-EXTENDED HERMITIAN FORM OVER $\mathbb{Z}[\mathbb{Z}]$

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Abstract. We describe a nonsingular hermitian form of rank 4 over the group ring $\mathbb{Z}[\mathbb{Z}]$ which is not extended from the integers. Moreover, we show that under certain indefiniteness asumptions, every nonsingular hermitian form on a free $\mathbb{Z}[\mathbb{Z}]$-module is extended from the integers. As a corollary, there exists a closed oriented 4-dimensional manifold with fundamental group $\mathbb{Z}$ which is not the connected sum of $S^1 \times S^3$ with a simply-connected 4-manifold.

## 1. Introduction

Let $A := \mathbb{Z}[x, x^{-1}]$ be the group ring of the infinite cyclic group (generated by the element $x$) which is equipped with the involution $x \mapsto x^{-1}$. With $t := x + x^{-1}$, the matrix

$$L := \begin{bmatrix} 1+t+t^2 & t+t^2 & 1+t & t \\ t+t^2 & 1+t+t^2 & t & 1+t \\ 1+t & t & 2 & 0 \\ t & 1+t & 0 & 2 \end{bmatrix}$$

describes a nonsingular hermitian form (extended from $\mathbb{Z}[t]$) on a free $A$-module of rank 4. We will prove the following

**Theorem 1.** *$L$ is not extended from the integers.*

Surprisingly, this seems to be the first example of such a form. ¿From the algebraic point of view, it is a classical problem to decide which forms over the polynomial ring $R[t]$ are extended from the ground ring $R$. There are however new difficulties in extending to the ring $R[x, x^{-1}]$. For example, over $\mathbb{Z}[t]$ the indefinite form $\left[\begin{smallmatrix} t & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is not extended from the integers (consider the values $t = 0, 1$ giving different types), but its extension to $A$ is extended from $\mathbb{Z}$ since $t = x + x^{-1}$. We found the form $L$ in [4], [5, p.474] as the first member of a family of forms $L_m$ of rank $4m$ over $\mathbb{Z}[t]$. In his paper, Quebbemann shows that for $m \geq 2$ the $L_m$ are not extended from the integers. His argument uses the evaluation maps

$$\mathbb{Z}[t] \xrightarrow{ev(z)} \mathbb{Z}, \quad z \in \mathbb{Z}$$

for $z = 0, 1$. He shows that $L_m(0)$ is the standard form whereas $L_m(1)$ is the (non-standard but well known) definite form $\Gamma_{4m}$ and thus $L_m$ cannot be extended.

The first author would like to thank the Max Planck Institut für Mathematik in Bonn for hospitality and support. The second author was supported by a Feodor Lynen fellowship from the Humboldt Foundation.

This argument breaks down for the group ring $A$ because the evaluation maps extend to $A$ only for $t = \pm 2$. Moreover, for $L = L_1$ all evaluation maps $ev(z)$ lead to the 4-dimensional standard form. Indeed, since all $L(z)$ have rank 4 and odd type it is enough to check this over the real numbers $\mathbb{R}$, where it follows from the existence of the homotopy $L = L(t)$.

The following result shows that under certain indefiniteness assumptions, a form over $A$ is indeed extended from the integers. Let $\epsilon : A \to \mathbb{Z}$ be the ring homomorphism which sends $x^{\pm 1}$ to 1.

**Theorem 2.** *Let $h$ be a nonsingular hermitian form on a free $A$-module of rank $r$ and denote by $s$ the signature of the form $\epsilon(h)$ on $\mathbb{Z}^r$. If $r - |s| \geq 6$ then $h$ is induced from the form $\epsilon(h)$.*

It is attractive to speculate that the same result holds under the weakest possible indefiniteness assumption $r > |s|$.

From our topological point of view, the question whether a form is extended form the integers arose from the study of closed oriented 4-manifolds with fundamental group $\mathbb{Z}$. For example, one knows that any nonsingular hermitian form on a finitely generated free $A$-module can be realized as the intersection form on $\pi_2$ of such a manifold $M$. Moreover, there are at most two 4-manifolds realizing a given form, see [2] for the precise formulation. Comparing this classification with the one for simply connected 4-manifolds [2], we obtain the following corollary from our two theorems above.

**Corollary .** *(1) There exists a closed oriented topological 4-manifold with infinite cyclic fundamental group which is not homotopy equivalent to the connected sum of $S^1 \times S^3$ with a closed simply-connected 4-manifold.*
*(2) If $M$ is a closed oriented topological 4-manifold with infinite cyclic fundamental group and with $b_2(M) - |\sigma(M)| \geq 6$ then $M$ is homeomorphic to the connected sum of $S^1 \times S^3$ with a unique closed simply-connected 4-manifold. In particular, $M$ is determined up to homeomorphism by its second Betti number $b_2(M)$, its signature $\sigma(M)$, and its Kirby-Siebenmann invariant.*

Note that this contradicts [3, Thm. 1.1].

## 2. $L$ is not extended

We briefly recall the notion of extended forms. We restrict ourselves to the case of forms over finitely generated free modules where one can describe forms by matrices. For a ring $R$ with involution $^-$ one defines an involution on $n \times n$-matrices $M = (m_{i,j})$ by $\overline{M} := (\bar{m}_{j,i})$ and then a sesquilinear [hermitian] form on a free $R$-module of rank $n$ is given by such a matrix $M$ [with $\overline{M} = M$]. If $\varphi : R \to S$ is a homomorphism of rings with involution and $M$ describes a [hermitian] form over $R$ then $\varphi(M) := (\varphi(m_{i,j}))$ determines a [hermitian] form over $S$ which is called *extended* from $R$ (via $\varphi$). For example, for any ring $R$ with unit 1 there is a homomorphism of rings with involution

$\varphi : \mathbb{Z} \to R$ given by $\varphi(1) := 1$. A form over $R$ is said to be *extended from the integers* if it is extended from $\mathbb{Z}$ via this map. If $R = \mathbb{Z}[G]$ is a group ring with the involution $\bar{g} := g^{-1}$ for $g \in G$, one can further consider the augmentation homomorphism

$$\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$$

sending all $g \in G$ to 1.

It follows from $\epsilon \circ \varphi = \mathrm{id}_{\mathbb{Z}}$ that if a form $M$ over $\mathbb{Z}[G]$ is extended from the integers then $M$ must be extended from the form $\epsilon(M)$.

Let us now come back to the case $R = A = \mathbb{Z}[x, x^{-1}]$ (and $\bar{x} := x^{-1}$) with the form given by $L$ (extended from $\mathbb{Z}[t]$ to $A$ via the map $\varphi$ which sends $t$ to $x + \bar{x}$). One easily computes $\det(L) = 1$ and thus $L$ is in fact nonsingular. Applying the augmentation $\epsilon$ to $L$ shows that if $L$ was extended from the integers then it must be equivalent over $A$ to the standard form (given by the identity matrix $\mathbb{1}$). The fact that $\epsilon(L)$ is equivalent to $\mathbb{1}$ can be either seen by a direct calculation or by the argument given in the introduction which uses the homotopy $L(t)$. Note that $L$ contains a vector (namely the first basis vector) of length $1 + t + t^2$. Therefore, the following Lemma implies our Theorem.

**Lemma 1.** *The standard form on $A^n$ does not contain a vector of length $1 + t + t^2$.*

*Proof.* We first introduce the notation

$$|v| := \sum_{i=1}^{n} v_i \cdot \bar{v}_i$$

for the length (w.r.t. the standard form) of a vector $v = (v_i) \in A^n$. Now suppose that $|v| = 1 + t + t^2$ and note that if $v_i = \sum_j z_{i,j} \cdot x^j$ with integers $z_{i,j}$ then

$$3 + (x + x^{-1}) + (x^2 + x^{-2}) = 1 + t + t^2 = |v| = \sum_{i=1}^{n} (\sum_{j,k} z_{i,j} z_{i,k} \cdot x^{j-k}).$$

Looking at the coefficient at $x^0$ this implies that 3 is the sum of the squares of all $z_{i,j}$. Thus $v$ has all together precisely 3 non-vanishing coefficients which all must be equal to $\pm 1$. After possibly re-ordering the components of $v$ we are reduced to the following cases:

1. $v = (v_1, v_2, v_3, 0, \ldots, 0), v_i = \pm x^{j_i}$.
   One easily checks that $|v| = 3$ which is a contradiction.
2. $v = (v_1, v_2, 0, \ldots, 0), v_1 = \alpha x^a + \beta x^b, v_2 = \pm x^c$. ($\alpha, \beta$ are signs $\pm 1$ and $a, b, c \in \mathbb{Z}$.) Again one easily checks that $|v| = 3 + \alpha\beta(x^{a-b} + x^{b-a})$, a contradiction.
3. $v = (v_1, 0, \ldots, 0), v_1 = \alpha x^a + \beta x^b + \gamma x^c$. ($\alpha, \beta, \gamma$ are signs $\pm 1$ and $a < b < c$ are integers.) In this case one obtains

$$|v| = 3 + \alpha\beta(x^{b-a} + x^{a-b}) + \gamma\beta(x^{c-b} + x^{b-c}) + \gamma\alpha(x^{c-a} + x^{a-c}).$$

In order to avoid a direct contradiction we must have $b-a=c-b=1$ and thus $c-a=2$. This implies that $\gamma=\alpha$ and thus the coefficient of $(x+x^{-1})$ on the right hand side is $\pm 2$, a final contradiction.

## 3. Proof of Theorem 2

Let $h'$ denote the $A$-valued hermitian form induced from $\epsilon(h)$. We want to show that $h$ and $h'$ are isomorphic. We first observe that $h$ and $h'$ are forms on free $A$-modules of the same rank and signature. The idea of the proof is to first allow stabilization with hyperbolic forms and then cancel these additional forms using the indefiniteness assumption. The first step uses computations of Wall's $L$-groups and the second the Bass cancellation theorem. One has to observe that the categories of *quadratic* forms used by Wall and Bass agree for free modules, see [1, I (4.4)]. In our notation, a hermitian form $h$ comes from such a quadratic form if $h=q+\bar{q}$ for some sesquilinear form $q$. Moreover, on a free module over a group ring of a group without elements of order 2, the hermitian form $h$ completely determines its underlying quadratic form. Thus the proof of Theorem 2 splits naturally into two cases:

*Case 1:* $h$ is quadratic, i.e. $h=q+\bar{q}$.

In this case, $h'$ also determines a quadratic form and we get two elements of the quadratic surgery obstruction group $L_0(A)$. This group is given by the Shaneson splitting theorem as

$$L_0(A)\cong L_0(\mathbb{Z})\oplus L_3(\mathbb{Z})=L_0(\mathbb{Z})\cong\mathbb{Z}.$$

Therefore, the isomorphism classes of such forms (after stabilizing by *hyperbolic* forms on free modules) are distinguished just by the ordinary signature. Hence $h$ and $h'$ are stably isomorphic. Now we can apply the cancellation theorem of Bass [1, IV (3.6)] to conclude that $h\cong h'$. More precisely, in the notation of [1, IV (3.1)] we have $R=A=\mathbb{Z}[\mathbb{Z}]$ and thus the subring $R_0$ genererated by all norms $a\cdot\bar{a}$, is just $\mathbb{Z}[t]\subset A$: The equation

$$(x+1)(\bar{x}+1)-x\bar{x}-1=x+\bar{x}=t$$

shows that $t\in R_0$. It is clear that $R_0$ is contained in the fixed ring of the involution and that this fixed ring is generated by the elements $x^n+x^{-n}$. Finally, one shows by induction that these elements lie in fact in $\mathbb{Z}[t]$.

The maximal dimension $d$ of the ring $R_0=\mathbb{Z}[t]$ equals $\dim\operatorname{Spec}(\mathbb{Z}[t])=2$ because every prime ideal is the product of maximal ideals.

By our assumption and the classification of indefinite integral forms $h'$ splits off $3=d+1$ hyperbolic forms. Therefore, the assumptions in Bass' cancellation theorem are satisfied.

*Case 2:* $h$ is not quadratic.

Then we consider $h$ and $h'$ as elements in the *symmetric* surgery group $L^0(A)$ which is the Grothendieck group of nonsingular hermitian forms on f.g. free $A$-modules.

The Ranicki splitting theorem gives

$$L^0(A) \cong L^0(\mathbb{Z}) \oplus L^3(\mathbb{Z}) = L^0(\mathbb{Z}) \cong \mathbb{Z}.$$

Therefore, there exist *metabolic* forms $M$ and $M'$ of the same rank such that

$$h \perp m \cong h' \perp m'.$$

Recall that by definition a metabolic form has a free $R$-basis $\{e_i, f_i\}, i = 1, \ldots, n$ such that

$$m(e_i, e_j) = 0 \quad \text{ and } \quad m(e_i, f_j) = \delta_{i,j}.$$

The next step is to show that $h$ and $h'$ are in fact stably isomorphic. For this we need some preparations:

**Lemma 2.** *Over any ring $R$, a metabolic form $m$ as above is the orthogonal sum of rank 2 metabolic forms.*

*Proof.* Given the above basis $\{e_i, f_i\}$ we define a new basis $\{e_i, f_i'\}$ by

$$f_i' := f_i - \sum_{j>i} m(f_i, f_j) \cdot e_j.$$

Then one easily checks that the restrictions of $m$ to $\langle e_i, f_i' \rangle$ decompose $m$ into an orthogonal sum of rank 2 metabolic forms. □

For a ring $R$ with involution consider the Tate-group

$$T(R) := \{a \in R \mid a = \bar{a}\}/\langle a + \bar{a} \rangle$$

If $\lambda$ is a hermitian form on some $R$-module $M$, we may consider the group homomorphism

$$Sq(\lambda) : M \to T(R), \quad m \mapsto \lambda(m, m).$$

This map becomes a homomorphism of $R$-modules if we define the $R$-action on $T(R)$ by $a \mapsto r \cdot a \cdot \bar{r}, r \in R, a \in T(R)$. Note that if $M$ is a free $R$-module then $\lambda$ comes from a quadratic form if and only if $Sq(\lambda) = 0$.

**Lemma 3.** *Let $\lambda$ be a hermitian form on some $R$-module $X$ such that $Sq(\lambda)$ is surjective. Then for any metabolic form $m$ there is an isomorphism*

$$\lambda \perp m \cong \lambda \perp \textit{hyperbolic}$$

*Proof.* By Lemma 2 it is enough to consider the case where $m$ has a basis $\{e, f\}$ with $m(e, e) = 0, m(e, f) = 1$ and $m(f, f) =: -a$. By assumption, there is an element $x_0 \in X$ such that $\lambda(x_0, x_0) = a$ since one can always perform a base change $f \mapsto f + r \cdot e$. Define an automorphism $\Phi$ of the $R$-module $X \oplus R^2$ by the formulas

$$\Phi(x) := x - \lambda(x, x_0) \cdot e \quad \text{ for } x \in X, \quad \Phi(e) := e \text{ and } \Phi(f) := x_0 + f.$$

Then the form $\lambda \perp m$ restricted to $\Phi(X)$ is isomorphic (via $\Phi$) to $(X, \lambda)$ and restricted to $\Phi(R^2)$ it is hyperbolic: $(\lambda \perp m)(x_0+f, x_0+f) = \lambda(x_0, x_0)+m(f, f) = 0$. Moreover, these two subspaces are perpendicular which proves our claim. $\square$

For our group ring $A = \mathbb{Z}[\mathbb{Z}]$ we have $T(A) = \mathbb{Z}/2$, generated by 1. Since we are in the case where $h$ is not quadratic we know that $Sq(h) \neq 0$ is surjective. The same is true for $h'$ and thus by Lemma 3 we conclude that $h$ and $h'$ are stably isomorphic. We denote the common stabilized form by $\lambda$.

**Lemma 4.** *The restriction of $\lambda$ to the kernel of the $A$-homomorphism $Sq(\lambda)$ is quadratic.*

This Lemma finishes the proof of Theorem 2 because we can now apply Bass cancellation to this restriction. More precisely, by our indefiniteness asumption $\lambda$, and thus also its restriction to $K := \mathrm{Ker}(Sq(\lambda))$, contains enough hyperbolic planes in order to satisfy the assumptions of [1, IV (3.5)]. In this corollary Bass shows that the automorphisms of the hyperbolic summand together with all transvections act transitively on the set of hyperbolic pairs in $K$. But these transvections can be extended to automorphisms of the whole module and may be thus used to cancel all additional hyperbolic summands. $\square$

*Proof of Lemma 4.* First observe that if $K$ was a free $A$-module then we were done because $Sq(\lambda)$ vanishes on $K$ by definition. In general, the obstruction for $(K, \lambda)$ being quadratic lies in the Tate-group of hermitian forms modulo those of the form $q+\bar{q}$. Since this is a $\mathbb{F}_2$-vector space, one checks that $(K, \lambda)$ is a quadratic form if and only if $(K/2, \lambda)$ is a quadratic form over the ring $A/2 = \mathbb{F}_2[\mathbb{Z}]$. We claim that $K/2$ is a free $A/2$-module which implies our lemma: just observe that $\lambda$ is the stabilization of the form $h'$ by a hyperbolic form and is thus induced from the integers. From the classification of indefinite integral forms we may conclude that $\lambda$ is the orthogonal sum of standard forms $(\pm 1)$ with respect to some basis $\{b_1, \ldots, b_n\}$. In particular, $Sq(\lambda)(b_i) = 1$ for all $i$. This shows that $K/2 \cong (A/2)^{n-1} \oplus I$ where $I$ denotes the augmentation ideal in $\mathbb{F}_2[\mathbb{Z}]$. But since we are working with the infinite cyclic group (generated by $x$) we know that the augmentation ideal is a free module (generated by $x - 1$). $\square$

## References

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