Supersymmetric Euclidean field theories and generalized cohomology

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1 Introduction

This paper is a report on our work trying to relate 2-dimensional field theories and elliptic cohomology, a subject pioneered by Graeme Segal two decades ago [Se1]. The version of elliptic cohomology we have in mind is the *topological modular form theory* developed by Mike Hopkins and Haynes Miller [Ho] and featured in talks by Matt Ando and Mike Hopkins at this conference. The name for this cohomology theory is motivated by the fact that the cohomology ring of a point is rationally isomorphic to the ring of integral modular forms (for the full modular group $SL_2(\mathbb{Z})$). We will be interested in the *periodic* version of this cohomology theory, denoted $TMF^*(X)$, which is periodic of period 24^2. There is a graded ring homomorphism

$$TMF^*(\text{point}) \longrightarrow MF^* = \bigoplus_{n\in\mathbb{Z}} MF^n \tag{1}$$

where MF^n is the abelian group of *weak integral modular forms of weight* $-\frac{n}{2}$ (the adjective weak means that we only require the modular forms to be meromorphic at infinity; see Definition 9).

On some philosophical level, it is obvious that there should be a close relationship between elliptic cohomology and field theories. The first version of elliptic cohomology constructed by Landweber and Stong [La] was defined as the cohomology theory built from a genus known as the *universal elliptic* or *Ochanine genus*, which associates to a closed spin manifold M a (level 2) modular form $\varphi(M)$. Witten provided a physical interpretation of $\varphi(M)$ as the partition function of a (not yet rigorously defined) 2-dimensional field theory associated to M (see [Wi1]). There is an analogous story relating TMF and the *Witten genus* which associates to a closed string manifold M a modular form $W(M)$. As Witten explains in the same paper, $W(M)$ also has a heuristic interpretation as the partition function of a 2-dimensional field theory. This field theory is known as the *non-linear σ-model of M*.

Alas, two decades later, there is still no geometric interpretation of elliptic cohomology in field theoretic terms despite the efforts by many mathematicians; some more recent ones include [HK] and [BDR]. In this paper we give a conjectural description of the elliptic cohomology of a manifold X in terms of supersymmetric $2|1$-*dimensional Euclidean field theories over* X (see 3). We offer two kinds of evidence for our conjecture:

- we show that supersymmetric field theories of dimension $0|1$ (resp. $1|1$) over X lead to ordinary cohomology (resp. K-theory) of X (see Theorem 2);
- we show that the partition function of a supersymmetric $2|1$-dimensional Euclidean field theory is a weak integral modular form(Theorem 4).

In the following section, we will give precise statements of these results and our conjecture, deferring the definition of field theories to later sections. A detailed description of the content of the sections can be found at the end of that section.

2 Results and conjectures

Our definition of EFT's (which is short for *Euclidean field theories*) is unfortunately pretty involved, and while about half of this paper is devoted to explaining the definition, this is by no means a complete account. Fortunately, we can explain the relationship between EFT's and generalized cohomology theories without first defining EFT's, and this is what we will do in this section.

In Definition 41 we will define $d|\Delta$-dimensional Euclidean field theories of degree n (or central charge n). These field theories are *supersymmetric* for $\delta > 0$; the non-negative integer δ is the number of odd symmetries present in the theory. More generally, if X is a smooth manifold, we will define EFT's *over* X, which can be thought of as *families* of EFT's parametrized by X. A Euclidean field theory over X should be thought of as a *geometric object* over X; for example we will see in Proposition 56 that a closed n-form over X can be interpreted as a $0|1$-dimensional EFT of degree n over X (and vice versa). Results of Florin Dumitrescu [Du] can be interpreted as showing that a vector bundle with connection over X gives rise to a $1|1$-dimensional EFT over X of degree 0.

Like differential forms or vector bundles with connections, Euclidean field theories over X of the same dimension $d|\delta$ can be added and multiplied.

They can be *pulled back* via smooth maps; i.e., a smooth map $f\colon Y \to X$ determines a functor

$$f^*\colon d|\delta\text{-}\mathrm{EFT}^n(X) \longrightarrow d|\delta\text{-}\mathrm{EFT}^n(Y)$$

between categories (and these functors compose strictly, unlike the case of vector bundles where $(fg)^*E$ is isomorphic, but not equal to g^*f^*E). We call two field theories $E_0, E_1 \in d|\delta\text{-}\mathrm{EFT}^n(X)$ *concordant* if there exists a field theory $E' \in d|\delta\text{-}\mathrm{EFT}^n(X \times \mathbb{R})$ such that $\iota_t^* E' \cong E_t$ for $t = 0, 1$, where $\iota_t\colon X \to X \times \mathbb{R}$ is the inclusion map $x \mapsto (x, t)$. We observe that the equivalence relation concordance can be defined for geometric objects over manifolds for which *pull-backs* and *isomorphisms* make sense. We note that by Stokes' Theorem two closed n-forms on X are concordant if and only if they represent the same deRham cohomology class; two vector bundles with connections are concordant if and only if they are isomorphic as vector bundles (i.e., disregarding the connections). Passing from an EFT over X to its concordance class forgets the geometric information while retaining the topological information. We will write $d|\delta\text{-}\mathrm{EFT}^n[X]$ for the set of *concordance classes* of $d|\delta$-dimensional supersymmetric EFT's of degree n over X.

Theorem 2. *Let X be a smooth manifold. Then there are natural ring isomorphisms*

$$\begin{aligned} 0|1\text{-}\mathrm{EFT}^n(X) &\cong \begin{cases} \Omega^{ev}_{cl}(X;\mathbb{C}) & n \text{ even} \\ \Omega^{odd}(X;\mathbb{C}) & n \text{ odd} \end{cases} \\ 1|1\text{-}\mathrm{EFT}^n[X] &\cong K^n(X) \end{aligned}$$

where $\Omega^{ev}_{cl}(X;\mathbb{C})$ (resp. $\Omega^{odd}_{cl}(X;\mathbb{C})$) stands for the even (resp. odd) closed differential forms on X.

The statement about EFT's of dimension $0|1$ is joint work with Henning Hohnhold and Matthias Kreck [HKST]. It follows that

$$0|1\text{-}\mathrm{EFT}^n[X] \cong \begin{cases} H^{ev}(X;\mathbb{C}) & n \text{ even} \\ H^{odd}(X;\mathbb{C}) & n \text{ odd} \end{cases}$$

where $H^{ev}(X;\mathbb{C})$ (resp. $H^{odd}(X;\mathbb{C})$) stands for the direct sum of the even (resp. odd) cohomology groups of X.

We want to mention that via the above isomorphisms the Chern-character $\mathrm{Ch}\colon K^0(X) \to H^{ev}(X;\mathbb{C})$ corresponds to a homomorphism $1|1\text{-}\mathrm{EFT}^0[X] \to 0|1\text{-}\mathrm{EFT}^0[X]$ given by *product with the circle* (see **??**). This was proven in Fei Han's thesis [Ha].

Conjecture 3. *There are natural ring isomorphisms*

$$2|1\text{-}\mathrm{EFT}^n[X] \cong TMF^n(X)$$

At this point, we don't have a map relating these two rings, not even if X is just a point. We do have a strategy to show that $2|1\text{-}\mathrm{EFT}^n[X]$ is a cohomology theory. The following theorem is the reason for our expectation that $2|1\text{-}\mathrm{EFT}^*[X]$ agrees with $TMF^*(X)$ rather than some other cohomology theory.

Theorem 4. *Let E be a Euclidean field theory of dimension $2|1$ and degree n (i.e., $E \in 2|1\text{-}\mathrm{EFT}^n(\mathrm{pt})$ in the notation above). Then the* partition function *of E (see Definition 60) belongs to MF^n.*

We want to emphasize that *supersymmetry* is a crucial feature. Non-super symmetric field theories (i.e., theories of dimension $d|0$) don't seem to be interesting from an algebraic topology point of view. We'll show that concordance classes of $0|0$-dimensional EFT's are trivial (see paragraph following Lemma 53), and suspect the same holds for field theories of dimension $1|1$ and $2|1$.

Let us briefly summarize the content of the paper. Sections 2-6 are devoted to the definition of Euclidean field theories. Our Euclidean field theories over a manifold are elaborate variants of Segal's axioms for conformal field theories. In section 3 we start with Segal's definition, describe internal categories as a convenient language to axiomatize field theories, carefully construct the bordism category we will be working with and arrive at a preliminary definition of Euclidean field theory (Definition **??**). The next three sections add bells and whistles: in section 3.3 we define what a d-dimensional field theory is (Definition 31) by adding the *smoothness* requirement, in section 4 we define supersymmetric Euclidean field theories (Definition 41), and in section 5 we define field theories of non-zero degree and twisted field theories. Section 7.1 (resp. 7.2 resp. 7.3) outlines our results on field theories of dimension $0|1$ (resp. $1|1$ resp. $2|1$). The reader might find the more concrete discussion in these sections a helpful illustration of the more abstract sections discussing the axiomatics of field theories. Section 6 explains how to evaluate d-dimensional field theories on closed d manifolds.

2.1 Segal's definition of a conformal field theory

In this section we start with Graeme Segal's definition of a 2-dimensional conformal field theory and elaborate suitably to obtain the definition of a d-dimensional Euclidean field theory. Segal has proposed an axiomatic description of 2-dimensional conformal field theories in a preprint that widely circulated for a decade and a half (despite the "do not copy" advice on the front) before it was published as [Se2]. In the published version, Segal added a forword/postscript commenting on developments since the original manuscript was written in which he proposes the following definition of conformal field theories.

Definition 5. (Segal [Se2, Postscript to section 4]) A *2-dimensional conformal field theory* (H, U) consists of the following two pieces of data:

1. A functor $Y \mapsto H(Y)$ from the category of closed oriented smooth 1-manifolds to locally convex complete topological vector spaces, which takes disjoint unions to (projective) tensor products, and
2. For each oriented cobordism Σ, with conformal structure, from Y_0 to Y_1 a linear-trace class maps $U(\Sigma)\colon H(Y_0) \to H(Y_1)$, subject to
   (a) $U(\Sigma' \circ \Sigma) = U(\Sigma) \circ U(\Sigma')$ when cobordisms are composed, and
   (b) $U(\Sigma \amalg \Sigma') = U(\Sigma) \otimes U(\Sigma')$.
   (c) If $f\colon \Sigma \to \Sigma'$ is a conformal equivalence between conformal bordisms, the diagram
   $$\begin{tikzcd} H(Y_0) \arrow[r, "U(\Sigma)"] \arrow[d, "H(f_{|Y_0})"'] & H(Y_1) \arrow[d, "H(f_{|Y_0})"] \\ H(Y_0') \arrow[r, "U(\Sigma')"] & H(Y_1') \end{tikzcd} \tag{6}$$
   is commutative.

Furthermore, $U(\Sigma)$ must depend smoothly on the conformal structure of Σ.

Condition (c) is not explicitly mentioned in Segal's postscript to section 4, but it corresponds to identifying conformal surfaces with parametrized boundary in his bordisms category $\mathcal{C}$ if they are conformally equivalent relative boundary, which Segal does in the first paragraph of §4.

We note that the first condition implies that H sends the empty set (viewed as a closed 1-manifold which is the unit w.r.t. taking disjoint union) to the vector space $\mathbb{C}$ (the unit w.r.t. the tensor product). If Σ is a closed oriented bordism, we can interpret it as a bordism from $\emptyset$ to $\emptyset$, and hence

$$U(\Sigma) \in \mathrm{Hom}(H(\emptyset), H(\emptyset)) = \mathrm{Hom}(\mathbb{C}, \mathbb{C}) = \mathbb{C}.$$

Definition 7. The *partition function* of a conformal field theory (H, U) is the function

$$Z \colon \mathbb{R}^2_+ = \{\tau \in \mathbb{R}^2 = \mathbb{C} \mid \operatorname{im} \tau > 0\} \to \mathbb{C} \qquad \text{given by} \qquad \tau \mapsto U(T_\tau),$$

where $T_\tau := \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}1)$ is the torus obtained from $\mathbb{C}$ by dividing out the lattice $\mathbb{Z}\tau + \mathbb{Z}1 \subset \mathbb{C}$.

There are a lot of possible variations of Definition 5. For example, conformal structures on the bordisms could be replaced by other types of geometric structures; e.g., no geometric structure on bordisms leads to *topological field theories*, Riemannian metrics lead to *Riemannian field theories*, and *Euclidean structures* (i.e., flat Riemannian metrics) lead to what we like to call *Euclidean field theories*.

Remark 8. It seems convenient to label the various types of field theories by the type of geometry on the bordisms (in physics lingo the worldsheets). For us it is important to differentiate between the two types of field theories determined by a Riemannian structure and a Euclidean structure (= *flat* Riemannian metric), respectively. It seems best to us to call the corresponding field theories *Riemannian field theories* and *Euclidean field theories*, despite the fact that this clashes with common use in physics, where the adjective *Euclidean* is used to indicate that one is dealing with Riemannian metrics rather than Lorentzian metrics, while flatness is not usually implied.

Other variations of the theme include replacing 2-dimensional bordisms by d-dimensional bordisms to obtain *field theories of dimension d* or by super manifolds of dimension $d|\delta$, furnished with appropriate super versions of conformal, Riemannian or Euclidean structures to obtain *field theories of dimension $d|\delta$*. Another variation is to equip all manifold Y and bordisms Σ with compatible maps to a fixed smooth manifold X; we will refer to the resulting theories as *field theories over X*.

In the next subsection we will discuss why we consider *Euclidean* field theories of dimension $2|1$ to be the most promising flavor of field theories to relate to TMF. In the ensuing sections, we will elaborate Segal's axioms in the following ways:

- We will show in section 3.1 that *internal categories* provide a categorical framework for Segal's definition in the sense that a field theory in the sense of Definition 5amounts to a functor between internal categories.
- In section 3.2 we will define the internal Euclidean bordism category d-EB that is the domain category in our preliminary definition 21 of a Euclidean field theory.
- In section 3.3 we will incorporate a form of *smoothness* in our definition of Euclidean field theory that is stronger than Segal's and that in our previous paper [ST]. The new requirement is that the vector space $H(Y)$ depends smoothly on Y (see Definition 31).
- In section sec:susy we will will define *super symmetric* field theories (Definition 41).
- Our use of internal categories makes it possible to give a very concise definition of a field theory of a non-zero degree (aka central charge; see Definition **??**) in section 5.

2.2 Which kind of field theories are appropriate?

For any flavor of field theory one can ask whether concordance classes of that type of field theory define a generalized cohomology theory. In this section we want to address the more specific question which type of field theory we should consider if we hope that their concordance classes correspond to TMF-cohomology classes. Such a field theory of degree n should in particular determine an element in $TMF^n(\mathrm{pt})$ and hence via the homomorphism (1) a *weak integral modular form* of weight $-\frac{n}{2}$. In our approach, the basic connection between 2-dimensional field theories and modular forms should be provided by associating to a field theory its partition function. Hence it is crucial to look for field theories whose partition functions are in fact integral modular forms (or rather modular functions, since for now we are looking at field theories of degree 0). After recalling what an integral modular form is, we'll discuss the modularity properties of the partition functions for the kinds of field theories mentioned above.

Definition 9. A *modular form of weight* k is a holomorphic function $f\colon \mathbb{R}^2_+ \to \mathbb{C}$ with the transformation property

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau) \qquad \text{for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}).$$

In addition, f is required to be *holomorphic at* $i\infty$ in the following sense. The above transformation property for the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ implies $f(\tau + 1) = f(\tau)$, and hence f can be thought of as a holomorphic function of $q = e^{2\pi i\tau}$ in the punctured open disc D^2_0. The Laurent series expansion

$$f(q) = \sum_{n\in\mathbb{Z}} a_n q^n$$

is called the *q-expansion* of f. We note that $\tau \to i\infty$ corresponds to $q \to 0$, which motivates the terminology that $f(\tau)$ is *holomorphic at* $i\infty$ if $f(q)$ extends to a holomorphic function over the disc, i.e., if $a_n = 0$ for $n < 0$. There doesn't seem to be a standard terminology for holomorphic functions $f\colon \mathbb{R}^2_+ \to \mathbb{C}$ with the transformation property of a modular form such that $f(q)$ has at most a pole at 0. We will refer to such functions as *weak modular forms.* A (weak) modular form is called *integral* if the coefficients a_n are integers.

We should mention that this is a very low-tech characterization of integrality; there is a much more conceptual definition of integral modular forms as sections of certain line bundles over the stack of elliptic curves over arbitrary commutative rings. It is this high-tech definition that evolved into the definition of *topological* modular forms. It would be extremely interesting to see whether an integral modular form could be constructed from $2|1$-dimensional Euclidean field theories by directly associating to any elliptic curve over a commutative ring R an element of an appropriate rank one module over R. One might hope that such a construction can be embellished to produce a *topological modular form* which provides a lift of the integral modular form partition function provided by the theorem above. Unfortunately, our proof of the theorem is quite different: no elliptic curves appear (other than elliptic curves over $\mathbb{C}$ and their super analogues), and we only use the low-tech characterization of integrality by by using supersymmetry to show that the q-expansion of the partition function has integral coefficients.

Topological field theories. We note that since any two tori are *diffeomorphic*, the commutative diagram (6) implies that the partition function is constant.

Conformal field theories. As documented by our previous paper [ST], we used to think that (supersymmetric) conformal field theories are the most promising type of field theory to relate to TMF, since its partition function has the transformation properties of a modular function (modular form if we consider field theories of *degree* n that we'll define in § 5). This follows from diagram (6) and the fact that the tori T_τ, $T_{\tau'}$ associated to points τ, τ' of the upper half plane $\mathbb{R}^2_+$ are conformally equivalent if and only if τ and τ' are in the same orbit of the standard action of $SL_2(\mathbb{Z})$ on $\mathbb{R}^2_+$. Conformal field theories which are *holomorphic* (cf. [HK]) have in particular holomorphic partition functions.

While the partition functions of holomorphic conformal field theories are weak modular forms, there are two problems. One is that we don't see an argument showing *integrality* of these partition functions. The other problem is that there seem to be *too few* conformal field theories to obtain all modular forms in the image of $\Psi\colon TMF^*(\mathrm{pt}) \to MF^*$ as partition functions. For example, if M is a closed string manifold of dimension n, its Witten genus $W(M)$ is known to be in the image of Ψ (this follows from Thm. 6.25 and Cor. 6.23 of [Ho]). As explained in the introduction, $W(M)$ should be the partition function of the 'non-linear σ-model' of M, a field theory which so far has not been defined rigorously. However, it is clear from perturbative calculations that this field theory cannot be *conformal* unless M satisfies some very strong geometric hypothesis including being Ricci flat.

Quantum field theories. At first sight this seems to be a silly choice, since for $\tau, \tau' \in \mathbb{R}^2_+$ in the same orbit of the $SL_2(\mathbb{Z})$-action, the tori T_τ, $T_{\tau'}$ are not necessarily *isometric* and hence there is no reason to expect the partition function to transform as a modular function.

In the fall of 2006 Witten mentioned in a conversation that the partition function of a *supersymmetric* quantum field theory is invariant under the $SL_2(\mathbb{Z})$-action. We found a way to show this in the context of our geometric definition, thus leading to Theorem 62. Our previous version of this result, Theorem 3.30 of [ST] held only for *conformal* field theories. Our argument for the modular invariance uses our new axiom that the vector spaces $H(Y)$ depend smoothly on Y.

Euclidean field theories. A Riemannian field theory always gives a Euclidean field theory (by restricting to *flat* Riemannian manifolds). So, a Riemannian field theory is more complicated object than a Euclidean field theory. For example, a 2-dimensional Riemannian field theory can be evaluated on any closed surface Σ to get a number (which depends on the Riemannian metric on Σ), while a Euclidean field theory can only be evaluated on surfaces of genus one, since by the Gauss-Bonnet Theorem, only surfaces of genus one admit flat metrics. In particular, our Theorem 62 is *stronger* than the same statement for Riemannian field theories. Put another way, higher genus surfaces aren't involved when trying to relate field theories to modular forms. The same should be true for topological modular forms, since they are constructed via a sheaf of spectra over the stack of elliptic curves (which over $\mathbb{C}$ are tori with conformal structures).

Supersymmetric Euclidean field theories. As mentioned before, supersymmetry is crucial to ensure the desired properties (holomorphicity, transformation property, integrality) of the partition function of a 2-dimensional field theory. While we haven't defined yet what an Euclidean field theory of dimension $d|\delta$ is, we want to mention that for our purposes we want field theories with a minimum of supersymmetry, i.e., with δ as small as possible for given d. It turns out that for $d = 0, 1, 2$, the minimum value of $\delta > 0$ is $\delta = 1$. Moreover, there is a unique flavor of supersymmetric Euclidean field theories for $0|1$, $1|1$ and *two* flavors for $2|1$. The two flavors of $2|1$-dimensional theories can be distinguished by their partition function – one leads to holomorphic, the other to anti-holomorphic partition functions. In this paper we'll be interested only in the first type, and we refer to them when we talk about $2|1$-dimensional EFT's.

3 Euclidean field theories

3.1 Internal categories

We note that the data (H, U) in Segal's definition of a conformal field theory (Definition 5) can be interpreted as a pair of symmetric monoidal functors. Here H is a functor from the category of closed oriented smooth 1-manifolds to the category of locally convex topological vector spaces. The domain of the functor U is the category whose objects are conformal bordisms and whose morphisms are conformal equivalences between conformal bordisms. The range of U is the category whose objects are trace-class operators between complete locally convex topological vector spaces and whose morphisms are commutative squares like diagram (6). The monoidal structure on the domain categories of H and U is given by the disjoint union, on the range categories it is given by the tensor product.

Better yet, the two domain categories involved fit together to form an *internal category* in the category of symmetric monoidal categories. The same holds for the two range categories, and the pair (H, U) is a functor between these internal categories. It turns out that internal categories provide a convenient language not only for field theories a la Segal; rather, all refinements that we'll incorporate in the following sections fit into this framework. What changes is the *ambient* category which is the category of symmetric monoidal categories now, and will be replaced later by the category of symmetric monoidal categories which are fibered over the category of manifolds (in section 3.3) resp. super manifolds (in section 4).

Internal categories are described e.g. in section XII.1 of the second edition of Mac Lane's book [McL]. Unfortunately, his version of internal categories is too strict to define the internal bordism category we need as domain. A suitable weakened version of internal categories and functors are defined for example by Martins-Ferreira in [M] who calls them *pseudo categories*. Since (weak) internal categories are central for our description of field theories, we will describe them in detail. We start with the definition of an internal category in an ambient category A. Then we explain why this is too strict to define our internal bordism category and go on to show how this notion can be suitably weakened if the ambient category A is a strict 2-category. Throughout we will be working with a version of internal categories *without units* obtained by deleting all data and properties having to do with units.

Definition 10. (Internal Category) Let A be category with pull-backs (here A stands for *ambient*). An *internal category* (without units) in A consists of two objects $\mathsf{C}_0, \mathsf{C}_1 \in \mathsf{A}$ and three morphisms

$$s, t\colon \mathsf{C}_1 \longrightarrow \mathsf{C}_0 \qquad c\colon \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \longrightarrow \mathsf{C}_1$$

(source, target and composition) such that the following diagrams are commutative. They express the usual axioms for a category. The commutativity of

$$\begin{tikzcd} \mathsf{C}_1 \arrow[d, "t"'] & \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "c"] & \mathsf{C}_1 \arrow[d, "s"] \\ \mathsf{C}_0 & \mathsf{C}_1 \arrow[l, "t"'] \arrow[r, "s"] & \mathsf{C}_0 \end{tikzcd} \tag{11}$$

specifies source and target of a composition; the commutativity of the diagram

$$\begin{tikzcd} \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \arrow[r, "c\times 1"] \arrow[d, "1\times c"'] & \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \arrow[d, "c"] \\ \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \arrow[r, "c"] & \mathsf{C}_1 \end{tikzcd} \tag{12}$$

expresses the associativity of the multiplication c.

Definition 13. Following MacLane (§XII.1), a functor $f\colon \mathsf{C} \to \mathsf{D}$ between internal categories C, D in the same ambient category A is a pair of morphisms in A

$$f_0\colon \mathsf{C}_0 \longrightarrow \mathsf{D}_0 \qquad f_1\colon \mathsf{C}_1 \longrightarrow \mathsf{D}_1.$$

Thought of as describing the functor on objects resp. morphisms, they are required to make the obvious diagrams commutative:

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, "s"] \arrow[d, "f_1"'] & \mathsf{C}_0 \arrow[d, "f_0"] \\ \mathsf{D}_1 \arrow[r, "s"] & \mathsf{D}_0 \end{tikzcd} \qquad \begin{tikzcd} \mathsf{C}_1 \arrow[r, "t"] \arrow[d, "f_1"'] & \mathsf{C}_0 \arrow[d, "f_0"] \\ \mathsf{D}_1 \arrow[r, "t"] & \mathsf{D}_0 \end{tikzcd} \tag{14}$$

$$\begin{tikzcd} \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \arrow[r, "f_1\times f_1"] \arrow[d, "c_{\mathsf{C}}"'] & \mathsf{C}_1 \arrow[d, "c_{\mathsf{D}}"] \\ \mathsf{D}_1 \times_{\mathsf{D}_0} \mathsf{D}_1 \arrow[r, "f_1"] & \mathsf{D}_1 \end{tikzcd} \tag{15}$$

As mentioned before, we would like to regard Segal's pair (H, U) as a functor between internal categories where the ambient category A is the category of symmetric monoidal categories. However, this is not quite correct due to the lack of associativity of the internal bordism category. In geometric terms, the problem is that if Σ_i is a bordism from Y_i to Y_{i+1} for

$i = 1, 2, 3$, then $(\Sigma_3 \cup_{Y_3} \Sigma_2) \cup_{Y_2} \Sigma_1$ and $\Sigma_3 \cup_{Y_3} (\Sigma_2 \cup_{Y_2} \Sigma_1)$ are not strictly speaking *equal*, but only canonically conformally equivalent. In categorical terms, this means that the diagram (12) is not commutative; rather, the conformal equivalence between these bordisms is a morphism in the category C_1 whose objects are conformal bordisms. This depends functorially on $(\Sigma_3, \Sigma_2, \Sigma_1) \in \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1$ and hence it provides an invertible natural transformation α between the two functors of diagram (12)

$$\begin{tikzcd} \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \arrow[r, "c \times 1"] \arrow[d, "1 \times c"'] & \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \arrow[d, "c"] \\ \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \arrow[r, "c"'] \arrow[ur, Rightarrow, "\alpha", "\cong"'] & \mathsf{C}_1 \end{tikzcd} \tag{16}$$

The moral is that we should relax the associativity axiom of an internal category by replacing the assumption that the diagram above is commutative by the weaker assumption that the there is an invertible 2-morphism α between the two compositions. This of course requires that the ambient category A can be refined to be a strict 2-category (which happens in our case, with objects/morphisms/2-morphisms being symmetric monoidal categories/symmetric monoidal functors/symmetric monoidal natural transformations).

This motivates the following definition:

Definition 17. An *internal category in a strict* 2*-category* A consists of objects C_0, C_1, morphisms s, t, c of A as in definition 10 and a 2-morphism α as in diagram (16). It is required that the diagrams (11) are commutative.

The 2-morphism α is subject to a coherence condition. In order to specify this, it will be convenient to write α in the form

$$c(c \times 1) \overset{\alpha}{\Longrightarrow} c(1 \times c)$$

We note that the domain and codomain of the *associator* α are both morphisms from $\mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1$ to C_1 obtained from the *composition morphism* c by the two possible ways of bracketing the three inputs. We find this aspect more transparent if we write $(-_1 \circ -_2) \circ -_3$ instead of $c(c \times 1)$ and $-_1 \circ (-_2 \circ -_3)$ instead of $c(1 \times c)$. With this notation, we have

$$(-_1 \circ -_2) \circ -_3 \overset{\alpha}{\Longrightarrow} -_1 \circ (-_2 \circ -_3)$$

The associator α is required to satisfy the well-known *Pentagon identity* which is the commutativity of the following diagram of 2-morphisms

$$\begin{tikzcd}
((-_1 \circ -_2) \circ -_3) \circ -_4 \arrow[rr, Rightarrow, "\alpha\circ 1"] \arrow[d, Rightarrow, "\alpha"'] & & (-_1 \circ (-_2 \circ -_3)) \circ -_4 \arrow[d, Rightarrow, "\alpha"'] \\
(-_1 \circ -_2) \circ (-_3 \circ -_4) \arrow[dr, Rightarrow, "\alpha"'] & & -_1 \circ ((-_2 \circ -_3) \circ -_4) \arrow[dl, Rightarrow, "1\circ\alpha"] \\
& -_1 \circ (-_2 \circ (-_3 \circ -_4)) &
\end{tikzcd}$$

Here each vertex is a morphism from $\mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1$ to C_1.

Next we define functors between categories internal to a 2-category by weakening Definition 13.

Definition 18. (Functors between internal categories). Let C, D be internal categories in a strict 2-category A. Then a functor $f\colon \mathsf{C} \to \mathsf{D}$ is a triple $f = (f_0, f_1, f_2)$, where $f_0\colon \mathsf{C}_0 \to \mathsf{D}_0$, $f_1\colon \mathsf{C}_1 \to \mathsf{D}_1$ are morphisms, and f_2 is an invertible 2-morphism

$$\begin{tikzcd}
\mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \arrow[r, "c_{\mathsf{C}}"] \arrow[d, "f_1\times f_1"'] & \mathsf{C}_1 \arrow[d, "f_1"] \\
\mathsf{D}_1 \times_{\mathsf{D}_0} \mathsf{D}_1 \arrow[r, "c_{\mathsf{D}}"'] \arrow[ur, Rightarrow, "f_2", "\cong"'] & \mathsf{D}_1
\end{tikzcd}$$

As in Definition 13 we require that the diagrams (14) commute. The 2-morphism f_2 is subject to a compatibility condition expressed by the commutativity of the following diagram of 2-morphisms:

$$\begin{tikzcd}
(f_1(-_1) \circ f_1(-_2)) \circ f_1(-_3) \arrow[r, Rightarrow, "\alpha_{\mathsf{D}}"] \arrow[d, Rightarrow, "f_2\circ 1"'] & f_1(-_1) \circ (f_1(-_2) \circ f_1(-_3)) \arrow[d, Rightarrow, "1\circ f_2"] \\
f_1(-_1 \circ -_2) \circ f_1(-_3) \arrow[d, Rightarrow, "f_2"'] & f_1(-_1) \circ f_1(-_2 \circ -_3) \arrow[d, Rightarrow, "f_2"] \\
f_1((-_1 \circ -_2) \circ -_3) \arrow[r, Rightarrow, "\alpha_{\mathsf{C}}"'] & f_1(-_1 \circ (-_2 \circ -_3))
\end{tikzcd}$$

Here the vertices are morphisms from $\mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1$ to D_1.

3.2 The Euclidean bordism category d-EB

Now we are ready to define d-RB, the category of d-dimensional Riemannian bordisms and the subcategory d-EB of d-dimensional Euclidean bordisms. Both are categories internal to SymCat, the strict 2-category of symmetric monoidal categories. Before giving the formal definition, let us make some remarks that hopefully motivate the definition below. In the usual gluing process of d-dimensional bordisms, the two glued bordisms intersect in a closed $(d-1)$-dimensional manifold Y, the object of the bordism category which is the source (resp. target) of the bordisms to be glued. For producing a Riemannian structure on the glued bordism (actually, even for producing a smooth structure on it), it is better if the intersection is an open d-manifold on which the Riemannian structures are required to match. This suggests to refine the *objects* of the bordism category d-RB to be pairs (Y, Y^c), where Y is an open Riemannian d-manifold and $Y^c \subset Y$ (the *core* of Y) is a closed $(d-1)$-dimensional submanifold of Y. We think of Y as a *Riemannian neighborhood* or a *Riemannian thickening* of the $(d-1)$-dimensional core manifold Y^c (this core manifold is the only datum usually considered). We will assume that the complement $Y \smallsetminus Y^c$ is a disjoint union of the form $Y \smallsetminus Y^c = Y^+ \amalg Y^-$, such that Y^c is in the closure of Y^+ as well as Y^-. This decomposition will allow us to distinguish domain and range of a bordism. This is customarily controlled by comparing the given orientation of the closed manifold Y^c with the orientation induced by thinking of it as a part of the boundary of an oriented bordism Σ. Our notion makes it unnecessary to furnish our manifolds with orientations.

Our main goal here is to define the d-dimensional Euclidean bordism category d-EB. It seems best to define first the Riemannian bordism category d-RB and then d-EB as the variation where we insist that all Riemannian metrics are *flat*. The simple reason is that we want to provide pictures and it's harder to draw interesting pictures of flat surfaces (e.g., the flat torus doesn't embed in $\mathbb{R}^3$).

Definition 19. The *d-dimensional Euclidean bordism category* d-RB is the category internal to the strict 2-category SymCat of symmetric monoidal categories defined as follows. The internal category d-EB is obtained completely analogously by using Euclidean structures (= flat Riemannian metrics) instead of Riemannian metrics throughout.

The object category d-RB_0. The objects of the symmetric monoidal category d-RB_0 are pairs (Y, Y^c), where Y is d-dimensional Euclidean manifold (without boundary, but usually non-compact) and $Y^c \subset Y$ is a closed piecewise smooth submanifold of dimension $d-1$, which we call the *core* of Y. These pairs come equipped with a decomposition $Y \setminus Y^c = Y^+ \amalg Y^-$, where $Y^\pm \subset Y$ are disjoint open subsets whose closures contain Y^c and whose union is $Y \setminus Y^c$. This extra datum is suppressed in the notation. Below is a picture of an object of 2-RB_0.

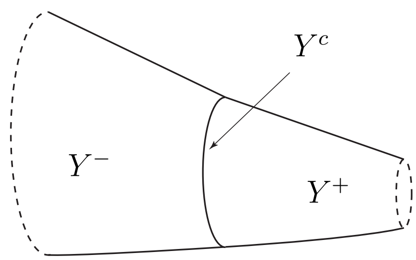

Figure 1: **An object of** 2-RB_0

A morphism from $Y = (Y, Y^c)$ to $Y' = (Y', (Y')^c)$ is the germ of an isometry $f\colon V \to V'$, where $V \subset Y$, $V' \subset Y'$ are open neighborhoods of Y^c resp. $(Y')^c$; these maps are required to send Y^c to $(Y')^c$ and $V^\pm \overset{\text{def}}{=} V \cap Y^\pm$ to $(V')^\pm \overset{\text{def}}{=} V' \cap (Y')^\pm$. As usual for germs, two such maps represent the same germ if they agree on some smaller open neighborhood of Y^c. Disjoint union gives d-RB_0 the structure of a symmetric monoidal category.

The morphism category d-RB_1 is defined as follows. An object of d-RB_1 consists of a pair $Y_0 = (Y_0, Y_0^c, Y_0^+, Y_0^-)$, $Y_1 = (Y_1, Y_1^c, Y_1^+, Y_1^-)$ of objects of d-RB_0 (the source resp. target) and a *Euclidean bordism* from Y_0 to Y_1, which is a triple (Σ, i_0, i_1) consisting of a Euclidean d-manifold Σ (without boundary) and isometric embeddings

$$i_0\colon W_0 \longrightarrow \Sigma \qquad \text{and} \qquad i_1\colon W_1 \longrightarrow \Sigma$$

with disjoint images. Here W_j is an open neighborhood of $Y_j^c \subset Y_j$. We require that the *core bordism* $\Sigma^c \overset{\text{def}}{=} \Sigma \setminus \big(i_0(W_0^+) \cup i_1(W_1^-)\big)$ is compact, where $W_j^\pm = W_j \cap Y_j^\pm$

Below is a picture of a Riemannian bordism; we usually draw the domain of the bordism to the right of its range, since we want to read compositions

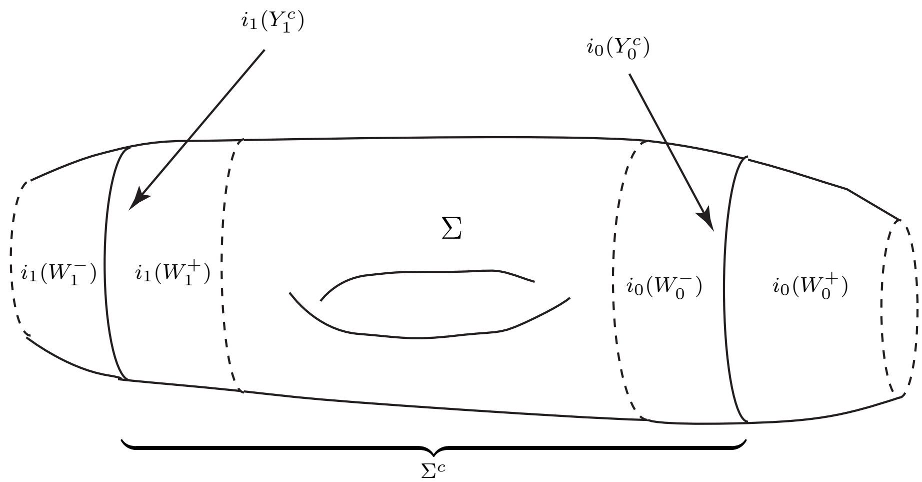

Figure 2: **A Riemannian bordism (object of** 2-RB_1**)**

of bordisms, like compositions of maps, from right to left. Roughly speaking, a bordism between objects Y_0 and Y_1 of d-RB_0 is just an ordinary bordism Σ^c from Y_0^c to Y_1^c equipped with a Riemannian metric, thickened up a little bit at its boundary to make it possible to glue two of them.

A morphism from a bordism Σ to a bordism Σ' is a germ of a triple of isometries

$$F\colon X \to X' \qquad f_0\colon V_0 \to V_0' \qquad f_1\colon V_1 \to V_1'.$$

Here X (resp. V_0 resp. V_1) is an open neighborhood of $\Sigma^c \subset \Sigma$ (resp. $Y_0^c \subset W_0 \cap i_0^{-1}(X)$ resp. $Y_1^c \subset W_1 \cap i_1^{-1}(X)$) and similarly for X', V_0', V_1'. We require the conditions for f_j to be a morphism from Y_j to Y_j' in d-RB_0, namely $f_j(Y_j^c) = (Y_j')^c$ and $f_j(V_j^{\pm}) = (V_j')^{\pm}$. In addition, we require that these isometrie are compatible in the sense that the diagram

$$\begin{tikzcd} V_1 \arrow[r, "i_1"] \arrow[d, "f_1"'] & X \arrow[d, "F"'] & V_0 \arrow[l, "i_0"'] \arrow[d, "f_0"'] \\ V_1' \arrow[r, "i_1'"'] & X' & V_0' \arrow[l, "i_0'"] \end{tikzcd}$$

is commutative. Two such triples (F, f_0, f_1) and (G, g_0, g_1) represent the same germ if there there are smaller open neighborhoods X'' of $\Sigma^c \subset X$ and

V_j'' of $Y_j \subset V_j \cap i_j^{-1}(X'')$ such that F and G agree on X'', and f_j and g_j agree on V_j'' for $j = 0, 1$.

Source, target and composition functors. There are obvious forgetful functors $s, t\colon d\text{-}\mathsf{RB}_1 \to d\text{-}\mathsf{RB}_0$ which send a bordism Σ from Y_0 to Y_1 to Y_0 resp. Y_1. These functors are compatible with taking disjoint unions and hence they are symmetric monoidal functors, i.e., morphisms in SymCat.

There is also a *composition functor*

$$c\colon d\text{-}\mathsf{RB}_1 \times_{d\text{-}\mathsf{RB}_0} d\text{-}\mathsf{RB}_1 \longrightarrow d\text{-}\mathsf{RB}_1$$

given by gluing bordisms. Let us describe this carefully, since there is a subtlety involved here due to the need to adjust the size of the Riemannian neighborhood along which we glue. Let Y_0, Y_1, Y_2 be objects of $d\text{-}\mathsf{RB}_0$, and let Σ, Σ' be bordisms from Y_0 to Y_1 resp. from Y_1 to Y_2. These data involve in particular isometric embeddings

$$i_1\colon W_1 \longrightarrow \Sigma \qquad i_1'\colon W_1' \longrightarrow \Sigma',$$

where W_1, W_1' are open neighborhoods of $Y_1^c \subset Y_1$. We set $W_1'' \stackrel{\text{def}}{=} W_1 \cap W_1'$ and note that our conditions guarantee that i_1 (resp. i_1') restricts to an isometric embedding of $(W_1'')^+ \stackrel{\text{def}}{=} W_1'' \cap Y_1^+$ to Σ (resp. Σ') (we note that this is not necessarily true when restricting to W_1''). We use these isometries to glue Σ and Σ' along W_1'' to obtain Σ'' defined as follows:

$$\Sigma'' \stackrel{\text{def}}{=} \left(\Sigma' \setminus i_1'((W_1')^+ \setminus (W_1'')^+)\right) \cup_{W_1''} \left(\Sigma \setminus i_1(W_1^- \setminus (W_1'')^-)\right)$$

The isometric embeddings $i_0\colon W_0 \to \Sigma$ and $i_2\colon W_2 \to \Sigma'$ induce isometric embeddings $W_0 \to \Sigma''$, $W_2 \to \Sigma''$ satisfying our conditions. This makes Σ'' a bordism from Y_0 to Y_2.

As explained above (see Equation (16)), the composition functor c is not strictly associative, but there is a natural transformation α as in diagram (16) which satisfies the pentagon identity.

We note that the categories $d\text{-}\mathsf{RB}_0$ and $d\text{-}\mathsf{RB}_1$ are both *groupoids* (i.e., all morphisms are invertible).

Definition 20. The category TV of (complete locally convex) topological vector spaces internal to SymCat (the category of symmetric monoidal categories) is defined as follows.

the object category TV_0 is the category whose objects are complete locally convex topological vector spaces over $\mathbb{C}$ and whose morphisms are continuous linear maps. The completed projective tensor product gives TV_0 the structure of a symmetric monoidal category.

the morphism category TV_1 is the symmetric monoidal category whose objects are continuous linear maps $V_0 \to V_1$ and whose morphisms are commutative squares. It is a symmetric monoidal category via the projective tensor product.

There are obvious source, target, and composition functors

$$s\colon \mathsf{TV}_1 \longrightarrow \mathsf{TV}_0 \qquad t\colon \mathsf{TV}_1 \longrightarrow \mathsf{TV}_0 \qquad c\colon \mathsf{TV}_1 \times_{\mathsf{TV}_0} \mathsf{TV}_1 \longrightarrow \mathsf{TV}_1$$

which make TV an internal category in SymCat. This is a *strict* internal category in the sense that associativity holds on the nose (and not just up to natural transformations).

Now we are ready for a preliminary first definition of a d-dimensional Euclidean field theory, which will be modified by adding a smoothness condition in the next section.

Definition 21. (Preliminary!) A d-dimensional *Euclidean field theory* over a smooth manifold X is a functor

$$E\colon d\text{-}\mathsf{EB}(X) \longrightarrow \mathsf{TV}$$

of categories internal to the strict 2-category of symmetric monoidal categories.

3.3 Euclidean field theories

The only feature missing from the above definition is the requirement that E should be *smooth*. Heuristically, this means that the vector space $E(Y)$ associated to an object Y of the bordism category as well as the operator $E(\Sigma)$ associated to a bordism Σ should depend smoothly on Y resp. Σ. To make this precise, we replace the categories d-EB, TV by *family categories* $d\text{-}\mathsf{EB}^{fam}$, TV^{fam} whose objects and morphisms are *smooth families* (i.e., smooth bundles over some parametrizing manifold S) of the objects/morphisms of the original categories. Let us illustrate this for the category TV.

Definition 22. The internal category TV^{fam} consists of categories TV_0^{fam}, TV_1^{fam} and functors s, t, c, where

the object category TV_0^{fam} has as objects smooth vector bundles $V \to S$ over arbitrary smooth manifolds S whose fibers are complete locally convex topological vector spaces. A morphism from $V_0 \to S_0$ to $V_1 \to S_1$ is a smooth vector bundle map

$$\begin{tikzcd} V \arrow[r, "\widehat{f}"] \arrow[d] & W \arrow[d] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

the morphism category TV_1^{fam} has as objects vector bundle maps

$$\begin{tikzcd} V_0 \arrow[rr, "g"] \arrow[dr] & & V_1 \arrow[dl] \\ & S & \end{tikzcd}$$

Moreover, a morphism from $g \colon V_0 \to V_1$ to $h \colon W_0 \to W_1$ is a pair of vector bundle maps $(\widehat{f_0} \colon V_0 \to W_0, \widehat{f_1} \colon V_1 \to W_1)$ covering the same map $f \colon S \to T$ on the base such that the obvious square commutes.

We note that there are functors from the categories TV_0^{fam} and TV_1^{fam} to the category Man of smooth manifolds which send a vector bundle $V \to S$ (resp. a vector bundle map $g \colon V_0 \to V_1$ of vector bundles over S) to the base space S. These functors are symmetric monoidal functors as well as *Grothendieck fibrations* in the sense explained below (see Definitions 26 and 27). Hence TV_0^{fam} and TV_1^{fam} are objects in the strict 2-category $\mathsf{SymCat}/\mathsf{Man}$ of symmetric monoidal categories fibered over Man (see Definition 27). Moreover, the functors s, t, c described above are morphisms in $\mathsf{SymCat}/\mathsf{Man}$ thus making TV^{fam} a category internal to $\mathsf{SymCat}/\mathsf{Man}$.

An excellent reference for fibrations of categories is [Vi], but we briefly recall the definition for the convenience of the reader who is not familiar with this language. Before giving the formal definition, it might be useful to look again at the example of the functor

$$p \colon \mathsf{TV}_0^{fam} \longrightarrow \mathsf{Man} \tag{23}$$

which sends a vector bundle its base space. We note that if $W \to T$ is a smooth vector bundle, and $f\colon S \to T$ is a smooth map, then there is a pull-back vector bundle $f^*W \to S$, and a tautological vector bundle map $\phi\colon V = f^*W \to W$ which maps to f via the functor p. The vector bundle morphism ϕ has a universal property called *cartesian*, which more generally can be defined for any morphism $\phi\colon V \to W$ of a category V equipped with a functor $p\colon \mathsf{V} \to \mathsf{S}$ to another category S. In the following diagrams, an arrow going from an object V of V to an object S of S, written as $V \mapsto S$, will mean that $p(V) = S$. Furthermore, the commutativity of the diagram

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, mapsto] & W \arrow[d, mapsto] \\ S \arrow[r, "f"] & T \end{tikzcd} \tag{24}$$

will mean that $p(\phi) = f$.

Definition 25. Let V be a category over S. An arrow $\phi\colon V \to W$ of V is *cartesian* if for any arrow $\psi\colon U \to W$ in V and any arrow $g\colon p(U) \to p(V)$ in S with $p(\phi) \circ g = p(\psi)$, there exists a unique arrow $\theta\colon U \to V$ with $p(\theta) = g$ and $\phi \circ \theta = \psi$, as in the commutative diagram

$$\begin{tikzcd} U \arrow[drr, bend left, "\psi"] \arrow[dr, dashed, "\theta"'] \arrow[dd, mapsto] & & \\ & V \arrow[r, "\phi"'] \arrow[dd, mapsto] & W \arrow[dd, mapsto] \\ R \arrow[dr, "g"'] \arrow[drr, bend left, "h"] & & \\ & S \arrow[r, "f"'] & T \end{tikzcd}$$

If $\phi\colon V \to W$ is cartesian, we say that the diagram (24) is a *cartesian square.*

In our example of the forgetful functor $\mathsf{TV}_0^{fam} \to \mathsf{Man}$, the usual pullback of vector bundles provides us with many cartesian squares. More precisely, the functor p is a *Grothendieck fibration* which is defined as follows.

Definition 26. A functor $p\colon \mathsf{V} \to \mathsf{S}$ is a *fibration* or *Grothendieck fibration* if pull-backs exist: for every object $W \in \mathsf{V}$ and every arrow $f\colon S \to T = p(W)$ in S, there is a cartesian square

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, mapsto] & W \arrow[d, mapsto] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

A *fibered category over* S is a category V together with a functor $p\colon \mathsf{V} \to \mathsf{S}$ which is a fibration. If $p_{\mathsf{V}}\colon \mathsf{V} \to \mathsf{S}$ and $p_{\mathsf{W}}\colon \mathsf{W} \to \mathsf{S}$ are fibered categories over S, then a *morphism of fibered categories* $F\colon \mathsf{V} \to \mathsf{W}$ is a base preserving functor ($p_{\mathsf{V}} \circ F = p_{\mathsf{W}}$) that sends cartesian arrows to cartesian arrows.

There is also a notion of base-preserving natural transformation between two morphisms from V to W. These form the 2-morphisms of a strict 2-category whose objects are categories fibered over S and whose morphisms are defined above. We will use the notation Cat/S for this 2-category of categories fibered over S.

We observe that the categories TV_0^{fam}, TV_1^{fam} are objects in the 2-category $\mathsf{Cat}/\mathsf{Man}$ of categories fibered over Man and the functors s, t, c are morphisms. In other words, TV^{fam} is a category internal to $\mathsf{Cat}/\mathsf{Man}$ (since c is strictly associative, it is even internal to $\mathsf{Cat}/\mathsf{Man}$ viewed as a *category*).

So far we have ignored the symmetric monoidal structures on these categories. We note that TV_0^{fam} has a symmetric monoidal structure which on objects $V \to S$, $W \to T$ of TV_0^{fam} is given by the external projective tensor product $V \otimes W \to S \times T$, and similarly for TV_1^{fam}. We note that the forgetful functor $p\colon \mathsf{TV}_j^{fam} \to \mathsf{Man}$ for $j = 0, 1$ is a *symmetric monoidal fibration* defined as follows.

Definition 27. For $\mathsf{V}, \mathsf{S} \in \mathsf{SymCat}$, a functor $p\colon \mathsf{V} \to \mathsf{S}$ is a *symmetric monoidal fibration* if

1. p is a strict symmetric monoidal functor;
2. p is a fibration;
3. the tensor product of cartesian arrows in V is cartesian.

We write $\mathsf{SymCat}/\mathsf{S}$ for the strict 2-category of symmetric monoidal categories fibered over S.

Similar to the passage from TV (a category internal to SymCat) to its family version TV^{fam} (category internal to $\mathsf{SymCat}/\mathsf{Man}$) we can go from the Euclidean bordism category d-EB (internal to SymCat) to its family version d-EB^{fam} (internal to $\mathsf{SymCat}/\mathsf{Man}$). A precise description of the latter category is our next goal. Furthermore, we would like to describe this internal category in a way that will make it easy to construct the 'super version' in the following section. So far, we've introduced Euclidean structures as Riemannian structures which are flat. This in not a good point of view for trying to work out the super analog, since it is easier to generalize the notion of Euclidean structure to super manifolds than the notion of Riemannian structures (see Remark **??**). The reason is that there is the following alternative description of Euclidean structures.

Definition 28. Let $\mathbb{E}^d$ be the d-dimensional euclidean space, and let $\mathrm{Isom}(\mathbb{E}^d)$ be the isometry group of $\mathbb{E}^d$ (the *Euclidean group*, which is the semi-direct product of the translation group $\mathbb{R}^d$ and the orthogonal group $O(d)$). A *Euclidean structure* on a d-manifold Y is a maximal atlas consisting of *charts* which are diffeomorphisms

$$Y \supset U_i \xrightarrow[\cong]{\varphi_i} V_i \subset \mathbb{E}^d$$

between open subsets of Y and open subsets of $\mathbb{E}^d$ such that the U_i's cover Y and for all i, j the transition function

$$\mathbb{E}^d \supset \varphi_i(U_i \cap U_j) \xrightarrow{\varphi_j \circ \varphi_i^{-1}} \varphi_j(U_i \cap U_j) \subset \mathbb{E}^d$$

is given by the restriction of an element of the Euclidean group $\mathrm{Isom}(\mathbb{E}^d)$.

It is clear that such an atlas determines a flat Riemannian metric on Y by transporting the standard metric on $\mathbb{E}^d$ to U_i via the diffeomorphism φ_i. Conversely, a flat Riemannian metric can be used to manufacture such an atlas.

The following definition generalizes this point of view on Euclidean manifold to families.

Definition 29. A *family of d-dimensional Euclidean manifolds* is a smooth map $p\colon Y \to S$ together with a maximal atlas consisting of charts which are diffeomorphisms φ_i between open subsets of Y and open subsets of $S \times \mathbb{E}^d$ making the following diagram commutative:

$$\begin{tikzcd} Y \supset U_i \arrow[rr, "\varphi_i", "\cong"'] \arrow[dr, "p"'] & & V_i \subset S \times \mathbb{E}^d \arrow[dl, "p_1"] \\ & S & \end{tikzcd}$$

We require that the open sets U_i cover Y and that for all i, j the *transition function*

$$S \times \mathbb{E}^d \supset \varphi_i(U_i \cap U_j) \xrightarrow{\varphi_j \circ \varphi_i^{-1}} \varphi_j(U_i \cap U_j) \subset S \times \mathbb{E}^d$$

is of the form $(s,v) \mapsto (s, g(s)v)$, where $g\colon p(U_i \cap U_j) \to \mathrm{Isom}(\mathbb{E}^d)$ is a smooth map. We note that the conditions imply in particular that p is a *submersion*.

If $Y \to S$ and $Y' \to S'$ are two families of Euclidean manifolds, a *morphism* between them is a pair of maps $(f, \widehat{f})$ making the following diagram commutative:

$$\begin{tikzcd} Y \arrow[r, "\widehat{f}"] \arrow[d] & Y' \arrow[d] \\ S \arrow[r, "f"] & S' \end{tikzcd}$$

We require that $\widehat{f}$ restricted to each fiber is injective and that it preserves the fiberwise euclidean structure in the sense that locally (by using charts) $\widehat{f}$ is of the form $(s,v) \mapsto (s, g(s)v)$ for some smooth map g to $\mathrm{Isom}(\mathbb{E}^d)$.

Definition 30. An object of $d\text{-}\mathsf{EB}_0^{fam}$ is a triple (S, Y, Y^c), where S is a smooth manifold, $Y \to S$ is a family of d-dimensional Euclidean manifolds, and $Y^c \subset Y$ is a codimension one submanifold such that the restriction of p to Y^c is a *proper submersion*. We note that the properness assumption is a family version of our old assumption that Y^c is compact, since it reduces to that assumption for $S = \mathrm{pt}$. Also part of the data, but suppressed in the notation is the decomposition of $Y \smallsetminus Y^c$ as the disjoint union of two open subsets $Y^{\pm}$, both of which contain Y^c in their closure.

A morphism from (S, Y, Y^c) to $(S', Y', (Y')^c)$ is the germ of a fiberwise isometry $(f, \widehat{f})$ from $V \to S$ to $V' \to S'$ where $V \subset Y$ (resp. $V' \subset Y'$ are open neighborhoods of $Y^c \subset Y$ (resp. $(Y')^c \subset Y'$. These maps are required to send Y^c to $(Y')^c$ and $V^{\pm} \overset{\mathrm{def}}{=} V \cap Y^{\pm}$ to $(V')^{\pm} \overset{\mathrm{def}}{=} V' \cap (Y')^{\pm}$. As usual for germs, two such maps represent the same germ if they agree on some smaller neighborhood of Y^c.

The categories $d\text{-}\mathsf{EB}_j^{fam}$, $j = 0, 1$ have a symmetric monoidal structure given by the (external) disjoint union of bundles (if $E \to S$ and $F \to T$ are bundles, the external disjoint union is the bundle $E \widehat{\amalg} F \to S \times T$ whose fiber over $(s,t) \in S \times T$ is the disjoint union $E_s \amalg F_t$ of the fibers E_s, F_t). Then the forgetful functors $d\text{-}\mathsf{EB}_j^{fam} \to \mathsf{Man}$ are symmetric monoidal fibrations, which makes $d\text{-}\mathsf{EB}_j^{fam}$ objects of $\mathsf{SymCat}/\mathsf{Man}$. The functors s, t, c are morphisms and the associator is a 2-morphism in this 2-category. In other words, the family bordism category $d\text{-}\mathsf{EB}^{fam}$ is a category internal to $\mathsf{SymCat}/\mathsf{Man}$.

Definition 31. A *Euclidean field theory of dimension* d is a functor

$$E\colon d\text{-}\mathsf{RB}^{fam} \longrightarrow \mathsf{TV}^{fam}$$

of internal categories over $\mathsf{SymCat}/\mathsf{Man}$, the strict 2-category of symmetric monoidal categories fibered over the category of manifolds. If X is a smooth manifold, a *Euclidean field theory of dimension* d *over* X is a functor

$$E\colon d\text{-}\mathsf{RB}^{fam}(X) \longrightarrow \mathsf{TV}^{fam}$$

of internal categories over $\mathsf{SymCat}/\mathsf{Man}$. Here d-$\mathsf{EB}^{fam}(X)$ is the generalization of d-$\mathsf{EB}^{fam}(X)$ obtained by adding to the Euclidean structure a map to X as part of the data. More precisely, d-$\mathsf{EB}_0^{fam}(X)$ is the category of triples (S, Y, Y^c), where $Y \to S$ is a family of Euclidean d-manifolds equipped with a smooth map $Y \to X$. Similarly, the bordisms $\Sigma \to S$ which form the objects of d-$\mathsf{EB}_1^{fam}(X)$ come now equipped with a smooth map $\Sigma \to X$.

Other variants can be obtained by furnishing the manifolds in question with orientations or spin-structures. More generally, we could work with manifolds with B-structures for some fiber bundle $B \to BO$. A more striking variation would be to replace TV^{fam} be another internal category in $\mathsf{SymCat}/\mathsf{Man}$.

4 Supersymmetric Euclidean field theories

In this section we will define *supersymmetric Euclidean field theories* by replacing manifolds by super manifolds and Euclidean structures by super Euclidean structures in the definitions of the previous two sections. It is possible to define supersymmetric *quantum* field theories along these lines, but super Riemannian structures are harder to define (see Remark **??**) and for reasons explained in the introduction, our primary interest is in *Euclidean* field theories.

We will first give a rapid introduction to super manifolds (more details can be found e.g. in [DM]) and explain what we mean by a *super Euclidean structure* in Definition **??**. Then we define the super versions of our categories TV^{fam} and d-EB^{fam}. The idea is to look at families parametrized by super manifolds. From a categorical point of view, these categories will be internal to the strict 2-category $\mathsf{SymCat}/\mathsf{SMan}$ of symmetric monoidal categories fibered over the category SMan of super manifolds. We end this section with our definition of a supersymmetric EFT as a functor between these internal categories (Definition 41).

4.1 Super manifolds

The monoidal category of $\mathbb{Z}/2$-graded vector spaces and their tensor product can be made into a *symmetric* monoidal category in two different ways. It is usual to speak of the *symmetric monoidal category of $\mathbb{Z}/2$-graded vector spaces* if the braiding isomorphism $V \otimes W \to W \otimes V$ is given by $v \otimes w \mapsto w \otimes v$. Equipped with the *signed braiding isomorphism* $V \otimes W \to W \otimes V$ given by $v \otimes w \mapsto (-1)\deg(v)\deg(w) w \otimes v$, it is referred to as the symmetric monoidal category of *super vector spaces*. A monoidal object in this category is called a *super algebra*, which amounts to a $\mathbb{Z}/2$-graded algebra. A commutative monoidal object is called a *commutative super algebra*. Explicitly, it is a $\mathbb{Z}/2$-graded algebra such that for any homogeneous elements a, b we have $a \cdot b = (-1)^{\deg(a)\deg(b)} b \cdot a$.

Definition 32. A *super manifold M of dimension $p|q$* is a ringed space $(M_{red}, \mathcal{O}_M)$ consisting of a topological space M_{red} (called the *reduced manifold*) and a sheaf $\mathcal{O}_M$ (called the *structure sheaf*) of commutative super algebras locally isomorphic to $(\mathbb{R}^p, \mathcal{C}^\infty(\mathbb{R}^p) \otimes \Lambda[\theta_1, \dots, \theta_q])$. Here $\mathcal{C}^\infty(\mathbb{R}^p)$ is the sheaf of smooth complex valued functions on $\mathbb{R}^p$, and $\Lambda[\theta_1, \dots, \theta_q])$ is the exterior algebra generated by elements $\theta_1, \dots, \theta_q$ (which is equipped with a $\mathbb{Z}/2$-grading by declaring the elements θ_i to be odd). It is more customary to require that $\mathcal{O}_M$ is a sheaf of real algebras; in this paper we will always be dealing with a *structure sheaf of complex algebras* (these are called *cs-manifolds* in [DM]). Abusing language, the global sections of $\mathcal{O}_M$ are called *functions* on M; we will write $C^\infty(M)$ for the algebra of functions on M.

As explained by Deligne-Morgan in [DM, §2.1], the quotient sheaf $\mathcal{O}_M/J$, where J is the ideal generated by odd elements, can be interpreted as a sheaf of smooth functions on M_{red}, giving it a smooth structure. Morphisms between super manifolds are defined to be morphisms of ringed spaces.

Let N be a smooth p-manifold and $E \to M$ be a smooth complex vector bundle of dimension q. Then $\Pi E \stackrel{\text{def}}{=} (N, \mathcal{C}^\infty(\Lambda E^*))$ is an example of a super manifold of dimension $p|q$. Here $\mathcal{C}^\infty(\Lambda E^*))$ is the sheaf of sections of the exterior algebra bundle $\Lambda E^* = \bigoplus_{i=0}^q \Lambda^i(E^*)$ generated by E^*, the bundle dual to E; the Π in ΠE stands for *parity reversal*. We note that *every* super manifold is isomorphic to a super manifold constructed in this way (but not every morphism $\Pi E \to \Pi E'$ is induced by a vector bundle homomorphism $E \to E'$). We will be in particular interested in the super manifold $\Pi T N_{\mathbb{C}}$ associated to the complexified tangent bundle of N. We note that

$$C^\infty(\Pi T N_{\mathbb{C}}) = C^\infty(N, \Lambda T N^*_{\mathbb{C}}) = \Omega^*(N; \mathbb{C}), \tag{33}$$

where $\Omega^*(N;\mathbb{C})$ is the algebra of complex valued differential forms on N.

4.2 Super Euclidean spaces and manifolds

Next we define the super analogues of Euclidean spaces, Euclidean groups and Euclidean manifolds. Our definitions of *super Euclidean space* and *super Euclidean group* are modeled on the definitions of *super Minkowski space* and *super Poincaré group* in [DF, §1.1], [Fr, Lecture 3]. In the non-super case, it is usual to first define the d-dimensional Euclidean space $\mathbb{E}^d$ as the manifold $\mathbb{R}^d$ equipped with its standard Riemannian metric, and then to define the Euclidean group as the isometry group $\mathrm{Isom}(\mathbb{E}^d)$ of $\mathbb{E}^d$. Alternatively, in the spirit of Felix Klein's Erlangen Program, one could first define $\mathrm{Isom}(\mathbb{E}^d)$ as the semi-direct product of the translation- and rotation group, and then define as we did in Definition 28 a Euclidean structure on a d-manifold as a maximal atlas whose transition functions belong to $\mathrm{Isom}(\mathbb{E}^d)$. We follow Klein's path in the super case.

To define super Euclidean space, we need the following data:

V	a real vector space with an inner product
Δ	a complex spinor representation of $Spin(V)$
$\Gamma\colon \Delta^* \otimes \Delta^* \to V_{\mathbb{C}}$	a $Spin(V)$-equivariant, non-degenerate symmetric pairing

Here $V_{\mathbb{C}}$ is the complexification of V. A complex representation of $Spin(V) \subset \mathbb{C}\ell(V)^{ev}$ is a *spinor representation* if it extends to a module over $\mathbb{C}\ell(V)^{ev}$, the even part of the complex Clifford algebra generated by V.

The super manifold $V \times \Pi\Delta^*$ is the *super Euclidean space.* We note that this is the super manifold associated to the trivial complex vector bundle $V \times \Delta^* \to V$, and hence the algebra of functions on this super manifold is the exterior algebra (over $C^\infty(V)$) generated by the Δ-valued functions on V, which we can interpret as *spinors on V*.

The pairing Γ allows us to define a multiplication on the super manifold $V \times \Pi\Delta^*$ by

$$(V \times \Pi\Delta^*) \times (V \times \Pi\Delta^*) \longrightarrow V \times \Pi\Delta^* \tag{34}$$
$$(v_1, w_1), (v_2, w_2) \mapsto (v_1 + v_2 + \Gamma(w_1 \otimes w_2), w_1 + w_2), \tag{35}$$

which gives $V \times \Pi\Delta^*$ the structure of a super Lie group (see [DM, §2.10]). Here we describe the multiplication map in terms of the *functor of points approach* explained e.g. in [DM, SS2.8-2.9]: for any super manifold S the set X_s of *S-points* in another super manifold X consists of all morphisms $S \to X$. For example, an S-point of the super manifold $V \times \Pi\Delta^*$ amounts to a pair (v, w) with $v \in C^\infty(S)^{ev} \otimes V_{\mathbb{C}}$, $w \in C^\infty(S)^{odd} \otimes \Delta^*$ and $\bar{v}_{red} = v_{red}$ (where $v_{red} \in C^\infty(S_{red}) \otimes V_{\mathbb{C}}$ is the restriction of v to the reduced manifold, and $\bar{v}_{red}$ is its complex conjugate). A morphism of super manifolds $X \to Y$ induces maps $X_S \to Y_S$ between the S-points of X and Y, which are functorial in S. Conversely, any collection of maps $X_S \to Y_S$ which is functorial in S comes from a morphism $X \to Y$ (by Yoneda's lemma).

We note that the spinor group $Spin(V)$ acts on the super manifold $V \times \Pi\Delta^*$ by means of the double covering $Spin(V) \to SO(V)$ on V and the spinor representation on Δ^*. The assumption that the pairing Γ is $Spin(V)$-equivariant guarantees that this action is compatible with the (super) group structure we just defined. We define the *super Euclidean group* to be the semi-direct product $(V \times \Pi\Delta^*) \rtimes Spin(V)$. By construction, this super group acts on the super manifold $V \times \Pi\Delta^*$ (the *translation* subgroup $V \times \Pi\Delta^*$ acts by group multiplication on itself, and $Spin(V)$ acts as explained above).

Up to isomorphism, the inner product space V and hence the associated Euclidean group is determined by the dimension of V. By contrast, the isomorphism class of the data (V, Δ, Γ) is in general not determined by the pair $(d, \delta) = (\dim_{\mathbb{R}} V, \dim_{\mathbb{C}} \Delta)$. Still, we will use the notation

$$\mathbb{E}^{d|\delta} \overset{\text{def}}{=} V \times \Pi\Delta^* \qquad \text{super Euclidean space} \tag{36}$$
$$\mathrm{Isom}(\mathbb{E}^{d|\delta}) \overset{\text{def}}{=} (V \times \Pi\Delta^*) \rtimes Spin(V) \qquad \text{super Euclidean group} \tag{37}$$

If necessary, δ could be interpreted as the isomorphism class of the data (Δ, Γ). In this paper, we are only interested in the cases $(d, \delta) = (0, 1), (1, 1), (1, 2)$. As we will see, there is no ambiguity in the first two cases, and for $(1, 2)$ we explicitly choose one the two isomorphism classes. We note that $\mathbb{C}\ell_0^{ev} = \mathbb{C}\ell_1^{ev} = \mathbb{C}$ and hence there is only one module Δ (up to isomorphism) of any given dimension δ. For $d = \dim V = 0$, the homomorphism Γ is necessarily trivial; for $d = 1$, $\delta = 1$, the homomorphism Γ is determined (up to isomorphism of the pair (Δ, Γ)) by the requirement that Γ is non-degenerate.

For $d = 2$, $\delta = 1$, there are two non-isomorphic modules Δ over $\mathbb{C}\ell_2^{ev} = \mathbb{C} \oplus \mathbb{C}$. To describe them explicitly as representations of $Spin(V)$, $V = \mathbb{R}^2$ we identify $Spin(V)$ with S^1 by the isomorphism $\varphi\colon S^1 \to Spin(V)$ which is characterized by requiring that the image of $\varphi(z)$ under the projection map $Spin(V) \to SO(V)$ acts on $V = \mathbb{R}^2 = \mathbb{C}$ by multiplication by z^2. The irreducible complex representations of S^1 are parametrized by the integers. For $k \in \mathbb{Z}$ let us write $\mathbb{C}_k$ for the complex numbers equipped with an S^1-action such that $z \in S^1$ acts by multiplication by z^k. Then up to isomorphism $\Delta^* = \mathbb{C}_k$ for $k = \pm 1$ and the S^1-equivariant homomorphism

$$\Delta^* \otimes \Delta^* = \mathbb{C}_{2k} \xrightarrow{\Gamma} V_{\mathbb{C}} = \mathbb{C}_2 \oplus \mathbb{C}_{-2}$$

is given by the inclusion map into the first summand (for $k = 1$) respectively second summand (for $k = -1$). For reasons that will become clear later, we fix our choice of (Δ, Γ) to be given by $k = -1$.

Now we can extend our definition of *Euclidean structures* to super manifolds. We will be brief, since this is a straightforward extension of Definitions 28 and 29.

Definition 38. Suppose a triple (V, Δ, Γ) as above is fixed with $d = \dim_{\mathbb{R}} V$, $\delta = \dim_{\mathbb{C}} \Delta$. A (super) *Euclidean structure* on a super manifold Y of dimension $d|\delta$ is a maximal atlas of Y such that all transition functions belong to the super Euclidean group $\mathrm{Isom}(\mathbb{E}^{d|\delta})$. Similarly, extending Definition 29 to super manifolds we define *families of $d|\delta$-dimensional Euclidean manifolds* $Y \to S$ over some super manifold S. These are in particular submersions whose fibers are super manifolds of dimension $d|\delta$ equipped with euclidean structures.

4.3 The superfamily categories TV^{sfam} and $d|\delta$-EB^{sfam}

To define the category TV^{sfam} we need to replace smooth vector bundles over manifolds by smooth vector bundles over supermanifolds. We recall that a smooth vector bundle can be characterized in terms of its sheaf of sections. This description generalizes immediately to the following definition of vector bundles (with possibly infinite dimensional fibers) over a super manifold S.

Definition 39. Let V be a locally convex topological vector space equipped with a $\mathbb{Z}/2$-grading and let S be a super manifold of dimension $p|q$. A *vector bundle over S with fiber V* is a sheaf $\mathcal{V}$ of topological $\mathcal{O}_S$-supermodules which is locally isomorphic to $\mathcal{O}_S \otimes V$ (projective tensor product of locally convex vector spaces). We note that $\mathcal{O}_S$ is a sheaf of *Frechet algebras* since for every open subset $U \subset S_{red}$ the algebra $\mathcal{O}_S$ has a unique Frechet algebra structure [GS].

Replacing vector bundles over manifolds by vector bundles over super manifolds, we can now mimic Definition 22 to define the internal category TV^{sfam}. The object category TV_0^{sfam} is the category of vector bundles over super manifolds, and the morphism category TV^{sfam} is the category of vector bundle maps between vector bundles over the same supermanifold. There are obvious forgetful functors from these categories to SMan which make TV_0^{sfam} and TV_1^{sfam} categories fibered over SMan. Moreover, the graded tensor product makes them symmetric monoidal categories fibered over SMan. The source-, target-, and composition functor are compatible with these structures and hence TV^{sfam} is a category internal to the strict 2-category $\mathsf{SymCat}/\mathsf{SMan}$ of symmetric monoidal categories fibered over SMan.

Definition 40. The *Euclidean bordism category* $d|\delta$-EB^{sfam} is a category internal to $\mathsf{SymCat}/\mathsf{SMan}$. It is a generalization of the category d-EB^{fam} (Definition 30) obtained by replacing families of d-dimensional Euclidean manifolds parametrized by families of $d|\delta$-dimensional Euclidean super manifolds (see Definition 38). More generally, if X is a smooth manifold, $d|\delta$-$\mathsf{EB}_0^{sfam}(X)$ is built from families of Euclidean super manifolds equipped with smooth maps to X.

Definition 41. Let $\mathbb{E}^{d|\delta}$ be a fixed super Euclidean space. A *supersymmetric Euclidean field theory* (supersymmetric EFT) of dimension $d|\delta$ is a functor

$$d|\delta\text{-}\mathsf{EB}^{sfam} \longrightarrow \mathsf{TV}^{sfam}$$

of categories internal to the strict 2-category $\mathsf{SymCat}/\mathsf{SMan}$ of symmetric monoidal categories over SMan. If X is a smooth manifold, a *supersymmetric Euclidean field theory over X* is a functor

$$d|\delta\text{-}\mathsf{EB}^{sfam}(X) \longrightarrow \mathsf{TV}^{sfam}$$

of categories internal to $\mathsf{SymCat}/\mathsf{SMan}$.

5 Twisted field theories

In this section we will define field theories of non-zero degree, or – in physics lingo – non-zero central charge. More generally, we will define *twisted* field theories over a manifold X. As explained in the introduction we would like to think of field theories over X as representing cohomology classes for certain generalized cohomology theories. Sometimes it is *twisted* cohomology classes that play an important role, e.g., the Thom class of a vector bundle that is not orientable for the cohomology theory in question. We believe that the twisted field theories defined below (see Definition 47) represent twisted cohomology classes, which motivates our terminology. We will outline a proof of this for $d|\delta = 0|1$ and $1|1$.

We will describe Euclidean field theories of degree n, or more generally, *twisted Euclidean field theories* as natural transformations between functors (see Definition 47). More precisely, these are functors between internal categories; their domain is our internal bordism category $d|1\text{-}\mathsf{EB}^{sfam}$. So our first task is to describe what is meant by a natural transformation between such functors. Then we will construct the range category and outline the construction of the relevant functors which will allow us to define Euclidean field theories of degree n. We end the section by relating Euclidean field theories of degree zero to field theories as defined in section **??** and by comparing our definition with Segal's definition of conformal field theories with non-trivial central charge.

5.1 Natural transformations between internal functors

Definition 42. (Natural transformations) Let A be a category with pullbacks and let C, D be categories internal to A. If f, g are two internal functors $\mathsf{C} \to \mathsf{D}$, a *natural transformation* n from f to g is a morphism $n \in \mathsf{A}(\mathsf{C}_0, \mathsf{D}_1)$, $\mathsf{C}_0, \mathsf{D}_1 \in \mathsf{A}$ making the following diagrams commutative:

$$\begin{tikzcd} & \mathsf{C}_0 \arrow[dl, "g_0"'] \arrow[d, "n"] \arrow[dr, "f_0"] & \\ \mathsf{D}_0 & \mathsf{D}_1 \arrow[l, "t"'] \arrow[r, "s"] & \mathsf{D}_0 \end{tikzcd} \qquad \begin{tikzcd} \mathsf{C}_1 \arrow[r, "g_1 \times ns"] \arrow[d, "nt \times f_1"'] & \mathsf{D}_1 \times_{\mathsf{D}_0} \mathsf{D}_1 \arrow[d, "c_{\mathsf{D}}"] \\ \mathsf{D}_1 \times_{\mathsf{D}_0} \mathsf{D}_1 \arrow[r, "c_{\mathsf{D}}"] & \mathsf{D}_1 \end{tikzcd} \tag{43}$$

We note that the commutativity of the first diagram is needed in order to obtain the arrows $gt \times f_1$, $g_1 \times ns$ in the second diagram. If the ambient category A is the category of sets, then $f, g\colon \mathsf{C} \to \mathsf{D}$ are just functors between small categories and n amounts to a natural transformation from the functor f to the functor g; the first diagram expresses the fact that for an object $a \in \mathsf{C}_0$ the associated morphisms $n_a \in \mathsf{D}_1$ has domain $f_0(a)$ and range $g_0(a)$. The second diagram expresses the fact that for every morphism $h\colon a \to b$ the diagram

$$\begin{tikzcd}
f_0(a) \arrow[r, "n_a"] \arrow[d, "f_1(h)"'] & g_0(a) \arrow[d, "g_1(h)"] \\
f_0(b) \arrow[r, "n_b"] & g_0(b)
\end{tikzcd}$$

is commutative.

Now let us assume that A is a strict 2-category, that C, D are internal categories in A and that $f, g\colon \mathsf{C} \to \mathsf{D}$ are internal functors (as defined in Definitions 17 and 18). As we have done in those definitions with internal category and functor, we will weaken the notion of natural transformation by requiring the second diagram only to be commutative up to a specified 2-morphism.

Definition 44. Let $f, g\colon \mathsf{C} \to \mathsf{D}$ be internal functors between internal categories in a strict 2-category A. A *natural transformation* from $f = (f_0, f_1, f_2)$ to $g = (g_0, g_1, g_2)$ is a pair $n = (n_0, n_1)$, where $n_0\colon \mathsf{C}_0 \to D_1$ is a morphism, and n_1 is a 2-morphism:

$$\begin{tikzcd}
\mathsf{C}_1 \arrow[r, "n_0 t \times f_1"] \arrow[d, "g_1 \times n_0 s"'] & \mathsf{D}_1 \times_{\mathsf{D}_0} \mathsf{D}_1 \arrow[d, "c_{\mathsf{D}}"] \\
\mathsf{D}_1 \times_{\mathsf{D}_0} \mathsf{D}_1 \arrow[r, "c_{\mathsf{D}}"] \arrow[ur, Rightarrow, "n_1"] & \mathsf{D}_1
\end{tikzcd} \tag{45}$$

It is required that the first diagram of Definition 42 is commutative, and that the morphisms f_2, g_2, n_1 and α_D of D_1 satisfy a coherence condition expressed as the commutativity of an octagon of morphisms in D_1.

5.2 Twisted Euclidean field theories

Next we define the internal category that we will need as range category.

Definition 46. We define the internal category TA of *topological algebras* as follows.

TA_0 Is the category whose objects are *topological algebras*. A topological algebra is a monoid in the symmetric monoidal category TV of topological vector spaces (equipped with the projective tensor product); i.e., an object $A \in \mathsf{TV}$ together with an associative multiplication $A \otimes A \to A$. Morphisms are continuous algebra isomorphisms.

TA_1 is the category of bimodules over topological algebras. A bimodule is a triple (A_1, B, A_0), where A_0, A_1 are topological algebras, and B is an A_1-A_0-bimodule (i.e., an object $B \in \mathsf{TV}$ with a morphism $A_1 \otimes B \otimes A_0 \to B$ satisfying the usual conditions required for a module over an algebra). A morphism from (A_1, B, A_0) to (A_1', B', A_0') is a triple (f_1, g, f_0) consisting of isomorphisms $f_0 \colon A_0 \to A_0'$, $f_1 \colon A_1 \to A_1'$ of topological algebras and a morphism $f \colon B \to B'$ of topological vector spaces which is compatible with the left action of A_1 and the right action of A_0 (A_1 acts on B' via the algebra homomorphism $A_0 \to A_0'$, and similarly for A_0).

There are obvious source and target functors $s, t \colon \mathsf{TA}_1 \to \mathsf{TA}_0$ given by $s(A_1, B, A_0) = A_0$, $t(A_1, B, A_0) = A_1$, and a composition functor

$$c \colon \mathsf{TA}_1 \times_{\mathsf{TA}_0} \mathsf{TA}_1 \longrightarrow \mathsf{TA}_1 \qquad (A_2, B, A_1), (A_1, B', A_0) \mapsto (A_2, B \otimes_{A_1} B', A_0)$$

The two categories TV_0, TV_1, the functors s, t, c and the usual associator for tensor products define a category TA of *topological algebras* internal to the strict 2-category Cat. We can do better by noting that the tensor product (in the category TV) makes TV_0, TV_1 symmetric monoidal categories and that the functors s, t, c are symmetric monoidal functors. This gives TA the structure of a category internal to the strict 2-category SymCat of symmetric monoidal categories.

More generally, we have a family version TA^{sfam} of this internal category, where TA_0^{sfam} is the category of smooth bundles of topological algebras over super manifolds, and TA_1^{sfam} is a category whose objects are quadruples (S, A_1, B, A_0), where A_1, A_2 are smooth bundles of topological algebras over the super manifold S, and B is a bundle of A_1-A_0-bimodules over S. The obvious forgetful functors $\mathsf{TA}_i^{sfam} \to \mathsf{SMan}$ for $i = 0, 1$ make these categories Grothendieck fibered over the category SMan of super manifolds; the external tensor product gives them the structure of symmetric monoidal categories Grothendieck fibered over SMan.

Definition 47. Let X be a smooth manifold and let

$$T\colon d|\delta\text{-}\mathsf{EB}^{sfam}(X) \to \mathsf{TA}$$

be a functor from the category $d|\delta\text{-}\mathsf{EB}^{sfam}$ of (families of) Euclidean super manifolds of dimension $d|\delta$ with maps to X (see Definition 40) to the category TA^{sfam} of families of topological algebras. A *T-twisted Euclidean field theory* of dimension $d|\delta$ is a natural transformation

$$\begin{tikzcd} d|\delta\text{-}\mathsf{EB}^{sfam}(X) \arrow[rr, bend left, "T^0"] \arrow[rr, bend right, "T"'] \arrow[rr, phantom, "\Downarrow E"] & & \mathsf{TA}^{sfam} \end{tikzcd}$$

where T^0 is the *constant functor* that maps every object Y of $d|\delta\text{-}\mathsf{EB}^{sfam}(X)_0$ to the algebra $\mathbb{C} \in \mathsf{TA}_0^{sfam}$ (which is the unit in the symmetric monoidal category of TA_0). It maps every object Σ of $d|\delta\text{-}\mathsf{EB}^{sfam}(X)_1$ to $(A_1, B, A_0) \in \mathsf{TA}_1^{sfam}$, where $A_1 = A_0$ is the algebra $\mathbb{C}$ and B is also $\mathbb{C}$, but regarded now as $\mathbb{C}$-$\mathbb{C}$-bimodule (this is the monoidal unit in TA_1^{sfam}). The functors $T_i^0\colon d|\delta\text{-}\mathsf{EB}_i^{sfam}(X) \to \mathsf{TA}_i$ send every morphism to the identity morphism of the monoidal unit of TA_i.

Let us unravel this definition in order to relate our notion of *twisted field theory* to Segal's definition of a *weakly conformal field theory* based on a modular functor E [**?**, Definition 5.2]. We note that the *twisting functor*

$$T = (T_0, T_1)\colon d|\delta\text{-}\mathsf{EB}^{sfam}(X) \longrightarrow \mathsf{TA}^{sfam}$$

associates to a closed manifold Y of dimension $d-1|q$ (equipped with a neighborhood of dimension $d|\delta$ with Euclidean structure and a map to X) a topological algebra $T_0(Y)$. To a Euclidean bordism Σ of dimension $d|\delta$ from Y_0 to Y_1 the functor T associates the $T_0(Y_1)$-$T_0(Y_0)$-bimodule $T_1(\Sigma)$. Of course, T_0 and T_1 are functors defined on *families* of these objects, but it is better to ignore that aspect for the time being.

To understand the mathematical content of the natural transformation $E = (E_0, E_1)$, we use Definition 44 and Diagram (45) in the case

$$\mathsf{C} = d|\delta\text{-}\mathsf{EB}^{sfam}(X) \qquad \mathsf{D} = \mathsf{TV}^{sfam} \qquad f = T^0 \qquad g = T \qquad n = E$$

We see that a natural transformation E is a pair (E_0, E_1), where

- $E_0\colon d|\delta\text{-}\mathsf{EB}^{sfam}(X)_0 \to \mathsf{TV}_1$ is a functor; in particular, E_0 associates to each closed manifold Y of dimension $d-1|q$ an object $E_0(Y)$ of TV_1, i.e., $E_0(Y)$ is a bimodule. The commutative triangle (43) implies that $E_0(Y)$ is a left module over $tE_0(Y) = g_0(Y) = T_0(Y)$ and a right module over $sE_0(Y) = f_0(Y) = T_0^0(Y) = \mathbb{C}$; in other words, $E_0(Y)$ is just a left $T_0(Y)$-module.

- According to diagram (45) E_1 is a natural transformation, i.e., for every bordism Σ from Y_0 to Y_1 (this is an object of C_1) we have a morphism $E_1(\Sigma)$ in D_1 whose domain (resp. range) is the image of $\Sigma \in \mathsf{C}_1$ under the functors

  $$\mathsf{C}_1 \xrightarrow{g_1 \times n_1 s} \mathsf{D}_1 \times_{\mathsf{D}_0} \mathsf{D}_1 \xrightarrow{c_{\mathsf{D}}} \mathsf{D}_1 \quad \text{resp.} \quad \mathsf{C}_1 \xrightarrow{n_1 t \times f_1} \mathsf{D}_1 \times_{\mathsf{D}_0} \mathsf{D}_1 \xrightarrow{c_{\mathsf{D}}} \mathsf{D}_1 .$$

  More explicitly, $E_1(\Sigma)$ is a map of left $T_0(Y_1)$-modules

  $$E_1(\Sigma)\colon T_1(\Sigma) \otimes_{T_0(Y_0)} E_0(Y_0) \longrightarrow E_0(Y_1) \otimes_{T_0^0(Y_1)} T_1^0(\Sigma) \cong E_0(Y_1).$$

We note that if the twisting functor T is the constant functor T_0, then $E_0(Y)$ is just a topological vector space, and $E_1(\Sigma)$ is a continuous linear map $E_0(Y_0) \to E_0(Y_1)$. It turns out that the commutative octagon in the definition of a natural transformation (Definition 44) amounts to the condition

$$E_1(\Sigma' \circ \Sigma) = E_1(\Sigma') \circ E_1(\Sigma)$$

if Σ is a bordism from Y_0 to Y_1, and Σ' is a bordism from Y_1 to Y_2. This in turn implies that the diagram (**??**) is commutative (on the nose). For future reference, we summarize these consideration as follows.

Lemma 48. *The category of T_0-twisted Euclidean field theories over X is isomorphic to the category of $d|\delta$-dimensional Euclidean field theory over X as in Definition 41.*

5.3 Field theories of degree n

A Euclidean field theory of dimension $d|\delta$ and degree $n \in \mathbb{Z}$ over a smooth manifold X is a T_n-twisted EFT, where

$$T^n\colon d|\delta\text{-}\mathsf{EB}^{sfam}(X) \longrightarrow \mathsf{TA}^{sfam}$$

is a specific functor. There is more general functor

$$T^W \colon d|\delta\text{-}\mathsf{EB}^{sfam}(X) \longrightarrow \mathsf{TA}^{sfam}$$

associated to any $\mathbb{Z}/2$-graded vector bundle $W \to X$ equipped with a metric and a metric preserving connection. The functor T^n is defined to be T^W, where W is the trivial even (resp. odd) vector bundle of dimension $|n|$ if $n \geq 0$ (resp. $n \leq 0$).

Suppressing the family aspect for now, we need to associate to each closed Euclidean super manifold Y of dimension $d-1|q$ (as usual equipped with a Euclidean neighborhood of dimension $d|\delta$) a topological algebra $T_0^W(Y)$, and to a super Euclidean bordism Σ from Y_0 to Y_1 a bimodule $T_1^W(\Sigma)$ equipped with a right action of $T_0^W(Y_0)$ and a left action of $T_0^W(Y_0)$. In particular if Σ is closed, then $T_1^W(\Sigma)$ is just a $\mathbb{Z}/2$-graded complex vector space. In fact, it is a graded complex line, namely the *Pfaffian line* of the skew-adjoint Dirac operator

$$\not{D} \colon L^2(\Sigma_{red}; S_+ \otimes f^*TX) \longrightarrow L^2(\Sigma_{red}; S_- \otimes f^*TX)$$

on the reduced manifold Σ_{red}. Here $S = S_+ \oplus S_-$ is the spinor bundle on Σ_{red} (w.r.t. the metric and spin structure induced by the super Euclidean structure on Σ), and $f \colon \Sigma_{red} \to X$ is the restriction to $\Sigma_{red} \subset \Sigma$ of the map $\Sigma \to X$ that is part of Σ as object of $d|\delta\text{-}\mathsf{EB}^{sfam}(X)_1$ (but that we suppress in the notation). We recall that the Pfaffian line is defined as the top exterior power of the finite dimensional vector space given by the kernel of the Dirac operator.

If Σ is not closed, but say a bordism from the empty set to the object $Y = (Y, Y^c, Y^+, Y^-) \in d|\delta\text{-}\mathsf{EB}^{sfam}(X)$, then the kernel of $\not{D}$ is no longer finite dimensional. Rather, the restriction of harmonic spinors on Σ to Y gives a *Lagrangian subspace* $L_\Sigma \subset \mathfrak{H}(Y)$ of the space of germs of harmonic spinors defined in some neighborhood of Y^c. Here a Lagrangian subspace is a maximal isotropic subspace with respect to the symmetric bilinear form b on $\mathfrak{H}(Y)$ which occurs as the boundary term in the equation

$$\langle \not{D}\Phi, \Psi \rangle - \langle \not{D}\Psi, \Phi \rangle = b(\Phi_{|Y}, \Psi_{|Y}).$$

We define $T_0^W(Y)$ to be the Clifford algebra generated by the vector space $\mathfrak{H}(Y)$ equipped with the symmetric bilinear form b; the Frechet topology on $\mathfrak{H}(Y)$ induces a topology on this Clifford algebra. It is well-known that a

Lagrangian subspace of the vector space generating a Clifford algebra determines a *Fock space module* over this algebra (characterized by the property that it contains a one-dimensional subspace whose annihilator is the given Lagrangian subspace). If Σ has no closed components, we define $T_1^W(\Sigma)$ to be the Fock module for the Clifford algebra $T_0^W(Y)$ determined by the Lagrangian subspace of boundary values of harmonic spinors. If Σ does have closed components, we apply the above construction to what is left if we ignore all closed components, and then tensor with the Pfaffian lines determined by the closed components. It can be shown that this construction is compatible with gluing of bordisms, and hence leads to a functor from $d|1\text{-}\mathsf{EB}(X)$ to TA.

This construction can be generalized to families of super Euclidean $q|1$-manifolds $\Sigma \to S$ by considering the fiberwise Dirac operator for the reduced bundle $\Sigma_{red} \to S_{red}$. For example, if $\Sigma \to S$ is a family of closed manifolds, this gives a graded line bundle over S_{red}. The problem is that this bundle does not extend in a canonical way to a line bundle over $S \supset S_{red}$, and hence we do not obtain a functor from $q|1\text{-}\mathsf{EB}^{sfam}(X)$ to TA^{sfam} this way. The solution is to work with differential operators on the super manifolds themselves rather than the Dirac operator on their reduced manifolds.

5.4 Differential operators on super Euclidean space

Our goal is to construct for Euclidean super manifolds Σ of dimension $d|1$ natural differential operators

$$\not{D}\colon C^\infty(\Sigma) \longrightarrow C^\infty(\Sigma; \mathrm{Ber}_\Sigma);$$

or more generally,

$$\not{D}_W\colon C^\infty(\Sigma; W) \longrightarrow C^\infty(\Sigma; \mathrm{Ber}_\Sigma \otimes W)$$

for any vector bundle with connection $W \to \Sigma$. Here Ber_Σ is the *Berezinian line bundle* over Σ whose sections can be integrated over Σ (provided Σ_{red} comes equipped with an orientation), see [DM, Proposition 3.10.5]; Deligne-Morgan use the notation $\mathrm{Ber}(\Omega^1_\Sigma)$ for Ber_Σ. If Σ is a manifold of dimension $d|0$ (i.e., an ordinary manifold), then Ber_Σ is just the top exterior power of the cotangent bundle. In general, if $x_1, \dots, x_d, \theta_1, \dots, \theta_q$ are local coordinates (x_i's even, θ_i's odd), their differentials $dx_1, \dots, d\theta_q \in \Omega^1_\Sigma$ lead to a local section $[dx_1, \dots, dx_d, d\theta_1, \dots, d\theta_q]$ of Ber_Σ.

To define $\not{D}$ on a Euclidean super manifold of dimension $d|1$, it suffices to define it on the super Euclidean space $E^{d|1}$ and to show that it is invariant under the action of the super Euclidean group. So suppose W is vector bundle with connection over $E^{d|1}$. Then we define the W-twisted Dirac operator

$$\not{D}_W \colon C^\infty(E^{d|1}; \mathrm{Ber}_{E^{d|1}} \otimes W) \longrightarrow C^\infty(E^{d|1}; W)$$

for $d = 0, 1, 2$ as follows:

$$\begin{aligned}\not{D}_W([d\theta]f) &= \nabla_{\partial_\theta} f\\ \not{D}_W([dx, d\theta]f) &= \nabla_{\partial_\theta - i\theta\partial_x} f\\ \not{D}_W([dz, d\bar{z}, d\theta]f) &= \nabla_{\partial_z} \nabla_{\partial_\theta + \theta\partial_{\bar{z}}} f\end{aligned}$$

6 Evaluating EFT's on closed manifolds

In this section we discuss what type of information can be obtained from an $d|\delta$-dimensional QFT E by evaluating it on (families of) closed Euclidean super manifolds of dimension $d|\delta$. So let

$$E \colon d|\delta\text{-}\mathsf{EB}^{sfam}(X) \longrightarrow \mathsf{TV}^{sfam}$$

be a Euclidean field theory of dimension $d|\delta$ over a smooth manifold X. Let $p \colon \Sigma \to S$ be a family of Euclidean super manifolds of dimension $d|\delta$. Assuming that p is proper, we can interpret $p \colon \Sigma \to S$ as a family of bordisms parametrized by S from the empty set to itself, or, more formally, as an object of the category $d|\delta\text{-}\mathsf{EB}_1^{sfam}$. If $f \colon \Sigma \to X$ is a smooth map, then

$$(S \xleftarrow{p} \Sigma \xrightarrow{f} X) \tag{49}$$

is an object of the category $d|\delta\text{-}\mathsf{EB}^{sfam}(X)_1$ with source and target

$$s(S \xleftarrow{p} \Sigma \xrightarrow{f} X) = t(S \xleftarrow{p} \Sigma \xrightarrow{f} X) = (S \longleftarrow \emptyset \longrightarrow X).$$

Applying the functor

$$E_1 \colon d|\delta\text{-}\mathsf{EB}^{sfam}(X)_1 \to \mathsf{TV}_1^{sfam}$$

we obtain an object of TV_1^{sfam}, i.e., a vector bundle morphism $V_0 \to V_1$ of vector bundles over S. The vector bundles V_i are determined by the source and target of the bordism Σ. Since $V_0 = V_1 = E_0(S \leftarrow \emptyset \rightarrow X)$ is the trivial 1-dimensional vectorbundle over S, we can regard

$$E_1(S \xleftarrow{p} \Sigma \xrightarrow{f} X) \in C^\infty(S)$$

as a function on the super manifold S.

The following result is a very useful observation.

Proposition 50. *Let G be a super Lie group which acts by symmetries on the object* (49) *(i.e., G acts on Σ and S making the maps p and f equivariant – with the trivial action on X – and the action preserves the fiberwise Euclidean structure on Σ). Then the function $E_1(S \xleftarrow{p} \Sigma \xrightarrow{f} X)$ is G-invariant with respect to the induced action on $C^\infty(S)$.*

The reader might wonder what being *G-invariant* means if G is a super group. We provide an explanation with our proof of the statement. From a more abstract point of view we can regard the triples $(S \xleftarrow{p} \Sigma \xrightarrow{f} X)$ as objects of a *stack* over the Grothendieck site Man of smooth manifolds. Evaluating a Euclidean field theory E on such an objects gives a *smooth function on this stack*.

Proof. Let $\mu\colon G \times S \to S$ be the action map, and $p_2\colon G \times S \to S$ be the projection map. The G-action on Σ provides a bundle isomorphism preserving the fiberwise Euclidean structure between the pullback bundles

$$p_2^*\Sigma \to G \times S \qquad \text{and} \qquad \mu^*\Sigma \to G \times S$$

Moreover, the G-equivariance assumption on f guarantees that this isomorphism is compatible with the natural maps $p_2^*\Sigma \xrightarrow{f\widehat{p}_2} X$ and $\mu^*\Sigma \xrightarrow{f\widehat{\mu}} X$. In other words, the objects

$$(G \times S \leftarrow p_2^*\Sigma \xrightarrow{f\widehat{p}_2} X) \qquad \text{and} \qquad (G \times S \leftarrow \mu^*\Sigma \xrightarrow{f\widehat{\mu}} X)$$

are isomorphic in $d|\delta\text{-}\mathsf{EB}^{sfam}(X)_1$ via an isomorphism which is the identity on the parameter space $G \times S$. Compatibility of the functor E_1 with pullbacks then implies that the pullback functions p_2^*h and μ^*h agree for

$$h = E_1(S \xleftarrow{p} \Sigma \xrightarrow{f} X) \in C^\infty(S).$$

If G is a discrete group, the condition $p_2^*h = \mu^*h$ is obviously equivalent to the invariance of $h \in C^\infty(S)$. For a super Lie group G, we adopt this condition as the definition of *G-invariance*. $\square$

Now we want to extend the above discussion to *twisted* field theories over X. So let E be a $d|\delta$-dimensional T-twisted Euclidean field theory over X (see Definition 47). In the paragraphs following that definition we discussed what this means in more concrete terms: if Σ is a bordism from Y_0 to Y_1, then we have the following data:

- for $i = 0, 1$ we have a topological algebra $T_0(Y_i)$ and a left $T_0(Y_i)$-module $E_1(Y_i)$;
- We have a $T_0(Y_1)$-$T_0(Y_0)$-bimodule $T_1(\Sigma)$ and a $E_1(Y_1)$-linear map
$$E_2(\Sigma)\colon T_1(\Sigma) \otimes_{T_0(Y_0)} E_1(Y_0) \longrightarrow E_1(Y_1)$$

In particular, if $Y_0 = Y_1 = \emptyset$, then $T_0(Y_0) = T_0(Y_1) = \mathbb{C}$ (as algebras), $E_1(Y_0) = E_1(Y_1) = \mathbb{C}$ (as modules), and $E_2(\Sigma)$ can be identified with an element of the dual space $T_1(\Sigma)^*$. In the above discussion, we've suppressed the family aspect, as well as the map from the bordism Σ to the manifold X. Putting this back in, we see that given a family of closed bordisms

$$(\, S \xleftarrow{p} \Sigma \xrightarrow{f} X\,) \in d|\delta\text{-}\mathsf{EB}^{sfam}(X)_1 \tag{51}$$

we obtain

1. a vector bundle $T_1(\, S \xleftarrow{p} \Sigma \xrightarrow{f} X\,)$ over S;
2. a section of the dual bundle
$$E_2(\, S \xleftarrow{p} \Sigma \xrightarrow{f} X\,) \in C^\infty(S; T_1^*(\, S \xleftarrow{p} \Sigma \xrightarrow{f} X\,)).$$

Now let us assume that as in Proposition 50 a super Lie group G acts by symmetries on $(\, S \xleftarrow{p} \Sigma \xrightarrow{f} X\,)$. This implies that $T_1(\, S \xleftarrow{p} \Sigma \xrightarrow{f} X\,)$ is a G-equivariant vector bundle over S, and generalizing Proposition 50 we have the following result.

Proposition 52. *Let G be a super Lie group which acts by symmetries on $(\, S \xleftarrow{p} \Sigma \xrightarrow{f} X\,)$. Then $E_1(\, S \xleftarrow{p} \Sigma \xrightarrow{f} X\,)$ is a G-equivariant section of the vector bundle $T_1^*(\, S \xleftarrow{p} \Sigma \xrightarrow{f} X\,)$ over S.*

Interpreting the triples $(\, S \xleftarrow{p} \Sigma \xrightarrow{f} X\,)$ as objects of a stack over Man, the twist functor T determines a vector bundle over this stack (given by $(\, S \xleftarrow{p} \Sigma \xrightarrow{f} X\,) \mapsto T_1^*(\, S \xleftarrow{p} \Sigma \xrightarrow{f} X\,)$). The T-twisted field theory E determines a section of this bundle (given by $Q_1(\, S \xleftarrow{p} \Sigma \xrightarrow{f} X\,)$).

7 EFT's of low dimensions

7.1 EFT's of dimension $d = 0$

Let us consider a 0-dimensional Euclidean field theory E over a smooth manifold X in the sense of our Preliminary Definition 21; i.e., E is a functor

$$E\colon 0\text{-}\mathsf{EB}(X) \longrightarrow \mathsf{TV}$$

between categories internal to the strict 2-category of symmetric monoidal categories. These are very big words for the very simple situation we are looking at here:

- $0\text{-}\mathsf{EB}(X)_0$ is the category whose objects are -1-dimensional manifolds with maps to X; it has only one object, the empty set, and only the identity morphism. Since $E_0\colon 0\text{-}\mathsf{EB}(X)_0 \longrightarrow \mathsf{TV}_0$ is a symmetric monoidal functor, it must send $\emptyset$ to a vector space isomorphic to $\mathbb{C}$, the monoidal unit in TV_0.

- An object of $0\text{-}\mathsf{EB}(X)_1$ is a 0-manifold Σ with a map $f\colon \Sigma \to X$. Thinking of Σ as a bordism from $\emptyset$ to $\emptyset$, the functor $E_1\colon 0\text{-}\mathsf{EB}(X)_1 \to \mathsf{TV}_1$ associates to (Σ, f) a continuous linear map

  $$E_1(\Sigma, f) \in \mathrm{Hom}(E_0(\emptyset), E_0(\emptyset)) = \mathbb{C};$$

  (more formally, $E_1(\Sigma, f)$ is an object of TV_1, i.e., a continuous linear map; source and target are determined by $s(E_1(\Sigma, f)) = E_0(s(\Sigma, f)) = E_0(\emptyset)$ and $t(E_1(\Sigma, f)) = E_0(t(\Sigma, f)) = E_0(\emptyset)$). Thinking of (Σ, f) as an unordered collection $x_1, \dots, x_k$ of points of X, the fact that E_1 is monoidal implies

  $$E_1(x_1, \dots, x_k) = E_1(x_1) \cdot \cdots \cdot E_1(x_k) \in \mathbb{C}.$$

  In particular, the functor E_1 is determined by the function $X \to \mathbb{C}$ $x \mapsto E_1(x)$.

We conclude that a 0-dimensional Euclidean field theory E over X (in the sense of the preliminary Definition 31)determines a function $X \to \mathbb{C}$ (not necessarily smooth or even continuous) and conversely, such a function determines a 0-dimensional EFT E over X (up to the choice of $E_0(\emptyset)$, i.e., up to natural transformations).

Now let us discuss 0-dimensional EFT's over X in the sense of the proper Definition 31; i.e., as a functor

$$E\colon 0\text{-}\mathsf{EB}^{fam}(X) \longrightarrow \mathsf{TV}^{fam}.$$

We recall that $0\text{-}\mathsf{EB}^{fam}(X)$ and TV^{fam} are family versions of the internal categories discussed above; in particular, the monoidal categories $0\text{-}\mathsf{EB}^{fam}(X)_i$, TV_i^{fam}, $i = 0, 1$ are fibered over the category of manifolds. Still, there is no interesting information in the monoidal functor E_0; it sends $(S \leftarrow \emptyset \to X) \in 0\text{-}\mathsf{EB}^{fam}(X)_0$ to a trivial 1-dimensional vector bundle over the parameter space S.

As in the non-family situation discussed above, the interesting information is contained in the symmetric monoidal functor

$$E_1\colon 0\text{-}\mathsf{EB}^{fam}(X)_1 \longrightarrow \mathsf{TV}_1^{fam}.$$

If $(S \xleftarrow{p} \Sigma \xrightarrow{f} X)$ is an object of $0\text{-}\mathsf{EB}^{fam}(X)_1$ (i.e., p is a finite sheeted cover, and f is a smooth map), then $E_1(S \xleftarrow{p} \Sigma \xrightarrow{f} X)$ is strictly speaking an endomorphism of the trivial line bundle $E_0(S \longleftarrow \emptyset \longrightarrow X)$, but it can be identified with a smooth function on X. The requirement that E_1 is a functor of symmetric monoidal categories over Man means that E_1 is compatible with pullbacks of covers, and that disjoint union of covers corresponds to products of the associated functions.

We claim that E_1 is determined by the function

$$E_1(X \xleftarrow{1} X \xrightarrow{1} X) \in C^\infty(X; \mathbb{C}).$$

To see this, consider the function

$$h = E_1(S \xleftarrow{p} \Sigma \xrightarrow{f} X) \in C^\infty(S; \mathbb{C})$$

associated to a general object of $0\text{-}\mathsf{EB}^{fam}(X)_1$. If the cover p is one-sheeted, the object $(S \xleftarrow{p} \Sigma \xrightarrow{f} X)$ can be obtained as a pull-back of the object $(X \xleftarrow{1} X \xrightarrow{1} X)$, and hence the function h is determined for one-sheeted coverings. In general, the restriction of $\Sigma \to S$ to sufficiently small subsets $S' \subset S$ gives a trivial covering, i.e., a disjoint union of one-sheeted coverings and hence the restriction of h to S' is determined by the monoidal property. Let us summarize the result of these considerations:

Lemma 53. *The above construction provides an equivalence between the category of* 0*-dimensional EFT's over* X *and the discrete category whose objects are the smooth functions on* X*.*

We recall that a category is called *discrete* if the only morphisms are identity morphisms. The discussion above illustrates in the simplest example that extending functors to the family category is a way to ensure their smoothness. Still, the result of all this work is disappointing for a topologist, since any two smooth functions are concordant, and hence there is only *one* concordance class of 0-dimensional field theories over X. Fortunately, supersymmetry comes to the rescue in the sense that there are interesting concordance classes of *supersymmetric* 0-dimensional EFT's over X as we will see now.

So let E be a $0|1$-dimensional EFT over X (the dimension $0|1$ makes it clear that E is supersymmetric). According to Definition 41 E is a functor

$$E\colon 0|1\text{-}\mathsf{EB}^{sfam}(X) \longrightarrow \mathsf{TV}^{sfam}$$

of internal categories. We recall that $0|1\text{-}\mathsf{EB}^{sfam}(X)_i$, TV_i^{sfam}, $i = 0, 1$ are symmetric monoidal categories fibered over the category SMan of super manifolds. Again, E_0 is uninteresting since the empty set is the only super manifold of dimension $-1|0$. The functor E_0 sends $(S \leftarrow \emptyset \to X) \in 0|1\text{-}\mathsf{EB}^{sfam}(X)_0$ to a trivial 1-dimensional vector bundle over the super manifold S.

Now let us consider an object

$$(S \xleftarrow{p} \Sigma \xrightarrow{f} X) \in 0|1\text{-}\mathsf{EB}^{sfam}(X)_1;$$

here p is a bundle of super manifolds with fibers of dimension $0|1$ equipped with super Euclidean structures along the fibers. It image under the functor $E_1\colon 0|1\text{-}\mathsf{EB}^{sfam}(X)_1 \to \mathsf{TV}_1^{sfam}$ is an endomorphism of a 1-dimensional trivial line bundle over the super manifold S which again can be interpreted as a function on S. Now let us consider a particular object, namely

$$\left(\underline{\mathrm{map}}(\mathbb{R}^{0|1}, X) \xleftarrow{p_1} \underline{\mathrm{map}}(\mathbb{R}^{0|1}, X) \times \mathbb{R}^{0|1} \xrightarrow{ev} X\right) \in 0|1\text{-}\mathsf{EB}^{sfam}(X)_1. \quad (54)$$

Here $\underline{\mathrm{map}}(\mathbb{R}^{0|1}, X)$ is the super manifold of maps from $\mathbb{R}^{0|1}$ to X, p_1 is the projection onto the first factor, and ev is the evaluation map. In general, the internal hom $\underline{\mathrm{map}}(Y, X)$ between two super manifolds X, Y is only a generalized super manifold (i.e, a functor $\mathsf{SMan}^{op} \to \mathsf{Set}$), but as discussed in **??**, for $Y = \mathbb{R}^{0|1}$, it is represented by the super manifold $\Pi T X_{\mathbb{C}}$. The image of this object under E_1 is a function on $\underline{\mathrm{map}}(\mathbb{R}^{0|1}, X)$ that we denote by

$$\omega \in C^\infty(\underline{\mathrm{map}}(\mathbb{R}^{0|1}, X)) = C^\infty(\Pi T X_{\mathbb{C}}) = \Omega^*(X;\mathbb{C}).$$

We recall from **??** that the functions on $\Pi T X_{\mathbb{C}}$ can be identified with complex valued differential forms on X.

Lemma 55. *ω is a closed, even form.*

In a nutshell, this follows since the isometry group G of the super Euclidean space $\mathbb{R}^{0|1}$ acts on the object (54); i.e., the evaluation map is G-equivariant w.r.t. the induced action on $\underline{\mathrm{map}}(\mathbb{R}^{0|1}, X)$ and the trivial action on X. This implies that ω is in the fixed point set of the induced G-action on $C^\infty(\underline{\mathrm{map}}(\mathbb{R}^{0|1}, X))$. A calculation identifies the fixed point set with the subspace of closed even forms. This argument is useful in other situations, and we give a general form of it in the next subsection. In subsection **??** we'll return to provide more details for the argument above, as well as discussing the more general situation of EFT's of dimension $0|1$ with non-trivial degree.

The object (54) is the *universal* object of $0|1\text{-}\mathsf{EB}^{sfam}(X)_1$ in the same sense that the object $(X \xleftarrow{1} X \xrightarrow{1} X)$ was universal for $0\text{-}\mathsf{EB}^{fam}(X)$: every object is locally (after restricting to sufficiently small subsets of S) isomorphic to a disjoint union of pull-backs of this universal object. This implies that the functor E_1 is determined by the differential form ω, the value of E_1 on the universal object, which provides the crucial step for the following result.

Proposition 56. *The above construction provides an equivalence between the category of $0|1$-dimensional EFT's over X and the discrete category whose objects are elements of $\Omega^{ev}_{cl}(X,\mathbb{C})$ (closed even forms with values in $\mathbb{C}$).*

Stoke's Theorem implies that two closed n-forms are concordant if and only if they differ by an exact form. Hence we obtain the following corollary.

Corollary 57. *There is a bijection between concordance classes of $0|1$-dimensional EFT's over X of degree n and the elements of the cohomology group $H^n(X,\mathbb{C})$.*

Now let us return to the discussion of 0-dimensional field theories of the previous section. First we will provide a proof of Lemma 55. Let G be the

Euclidean super group of isometries of the Euclidean super space $\mathbb{R}^{0|1}$. By construction, G is the semi-direct product $\mathbb{R}^{0|1} \rtimes \mathbb{Z}/2$, where $\mathbb{R}^{0|1}$ acts on itself by translations, and $\mathbb{Z}/2$ ($= Spin(V)$ for the trivial vector space V) acts on $\mathbb{R}^{0|1}$ by $\theta \mapsto -\theta$. The super group G acts by symmetries on

$$\left(\underline{\mathrm{map}}(\mathbb{R}^{0|1}, X) \xleftarrow{p_1} \underline{\mathrm{map}}(\mathbb{R}^{0|1}, X) \times \mathbb{R}^{0|1} \xrightarrow{ev} X\right).$$

If E is a 0-EFT over X, Proposition 50 implies that E_1 maps the above object to a G-invariant function ω on $\underline{\mathrm{map}}(\mathbb{R}^{0|1}, X)$. As before we identify functions on $\underline{\mathrm{map}}(\mathbb{R}^{0|1}, X)$ with differential forms on X. A calculation shows that the $\mathbb{Z}/2$-factor of G acts on $\Omega^*(X;\mathbb{C})$ by multiplication by -1 on the odd forms and trivially on the even forms. This implies that ω is an *even* form.

To understand the $\mathbb{R}^{0|1}$-action on $\underline{\mathrm{map}}(\mathbb{R}^{0|1}, X)$, we note that a $\mathbb{R}^{0|1}$-action on any super manifold Y can be identified with an odd vector field on Y with square zero. This is the odd analogue of the statement that an $\mathbb{R}$-action on an ordinary manifold is determined by a vector field. A calculation shows that the odd vector field (= odd derivation of functions) corresponding to the $\mathbb{R}^{0|1}$-action on $\underline{\mathrm{map}}(\mathbb{R}^{0|1}, X)$ is the usual exterior differential d acting on $\Omega^*(X;\mathbb{C})$. In particular the $\mathbb{R}^{0|1}$-invariant functions on $\underline{\mathrm{map}}(\mathbb{R}^{0|1}, X)$ are the *closed* differential forms on X. This proves Lemma 55.

Next we will discuss $0|1$-dimensional EFT's over X of degree n. Applying Proposition 52 to the situation at hand, we see that if E is a $0|1$-EFT of degree n over X, then we obtain a G-equivariant section ω of the dual of the G-equivariant vector bundle $T_1^n(S \xleftarrow{p} \Sigma)$ over $S = \underline{\mathrm{map}}(\mathbb{R}^{0|1}, X)$ (we note that by construction of the degree n twist functor T^n, this vector bundle doesn't depend on the map $\Sigma \to X$). By the compatibility of T_1^n with pullbacks, the vector bundle is the pullback of $T_1^n(\mathrm{pt} \longleftarrow \mathbb{R}^{0|1})$ (a line bundle over the point pt), which in turn can be identified with the n-tensor power of the line $T_1^1(\mathrm{pt} \longleftarrow \mathbb{R}^{0|1})$. Choosing an identification of the latter line with $\mathbb{C}$ allows us to identify sections of $T_1^n(S \xleftarrow{p} \Sigma)$ again with differential forms on X. A calculation shows that $\mathbb{Z}/2 \subset G$ acts on the line $T_1^1(\mathrm{pt} \longleftarrow \mathbb{R}^{0|1})$ by multiplication by -1. This implies that G-equivariant sections of $T_1^n(S \xleftarrow{p} \Sigma)$ correspond to the closed even forms (for n even) resp. closed odd forms (for n odd). This implies the analog of Proposition 56 for field theories of degree n. In particular we obtain the following generalization of Corollary 57.

Corollary 58. *There is a bijection between concordance classes of* $0|1$*-dimensional EFT's over* X *of degree* n *and the elements of the even cohomology* $H^{ev}(X, \mathbb{C})$ *(the sum of the cohomology groups* $H^k(X; \mathbb{C})$ *for* k *even) if* n *is even. For* n *odd, there is a bijection with the odd cohomology* $H^{odd}(X, \mathbb{C})$ *(the sum of the cohomology groups* $H^k(X; \mathbb{C})$ *for* k *odd).*

Remark 59. It is possible to consider *topological* field theories of dimension $0|1$ instead of *Euclidean* field theories by not requiring any geometric structure on the bordisms. Our discussion above goes through with one change: we need to replace the *isometry group* of $\mathbb{R}^{0|1}$ by the larger *diffeomorphism group* $\mathrm{Diff}(\mathbb{R}^{0|1})$. A calculation shows that the elements of $C^\infty(\underline{\mathrm{map}}(\mathbb{R}^{0|1}, X)) = \Omega^*(X; \mathbb{C})$ that are $\mathrm{Diff}(\mathbb{R}^{0|1})$-invariant are the closed differential forms of degree zero. Hence a topological field theory of dimension $0|1$ over X amounts to a closed 0-form on X. More generally, topological field theories over X of dimension $0|1$ and degree n correspond to closed n-forms on X.

7.2 EFT's of dimension $d = 1$

7.3 EFT's of dimension $d = 2$

In this subsection we will discuss Euclidean field theories of dimension 2 and $2|1$. The most basic invariant of a 2-dimensional EFT E is its *partition function* which is defined by evaluating E on *closed* Euclidean manifolds as described in general in section 6. We note that by the Gauss-Bonnet Theorem the only closed surface that admits a Euclidean structure (i.e., a flat metric) is the torus. In fact, the moduli space of flat tori can be parametrized by the product $\mathbb{R}_+ \times \mathbb{R}^2_+$ of the positive real line $\mathbb{R}_+$ and the upper half plane $\mathbb{R}^2_+$ (consisting of $\tau \in \mathbb{R}^2 = \mathbb{C}$ with positive imaginary part). This parametrization is given by associating to $(\ell, \tau) \in \mathbb{R}_+ \times \mathbb{R}^2_+$ the flat torus

$$T_{\ell,\tau} := \mathbb{C}/\ell(\mathbb{Z}\tau + \mathbb{Z}1).$$

While every flat surface is isometric to some $T_{\ell,\tau}$, the map which sends (ℓ, τ) to the isometry class of $T_{\ell,\tau}$ is not injective. A torus $T_{\ell,\tau}$ is isometric to $T_{\ell',\tau'}$ if and only if (ℓ, τ) and (ℓ', τ') are in the same orbit of the $SL_2(\mathbb{Z})$-action on $\mathbb{R}_+ \times \mathbb{R}^2_+$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\ell, \tau) = \left(\ell|c\tau + d|, \frac{a\tau + b}{c\tau + d} \right)$$

If we forget about the first factor, the quotient of $\mathbb{R}^2_+$ modulo $SL_2(\mathbb{Z})$ has the well-known interpretation as the moduli space of *conformal structures* on tori. Similarly, the above statement means that the moduli space of *Euclidean structures* on tori can be identified with the quotient of $\mathbb{R}_+ \times \mathbb{R}^2_+$ modulo $SL_2(\mathbb{Z})$. As in the conformal situation, the product $\mathbb{R}_+ \times \mathbb{R}^2_+$ itself can be interpreted as the moduli space of Euclidean tori furnished with a basis for their integral first homology. Then the $SL_2(\mathbb{Z})$-action above corresponds to changing the basis.

Definition 60. Let E be a 2-dimensional Euclidean field theory. Then its *partition function*

$$Z_E \colon \mathbb{R}_+ \times \mathbb{R}^2_+ \longrightarrow \mathbb{C}$$

is defined by $Z_E(\ell, \tau) = E_1(T_{\ell,\tau})$, i.e., by evaluating the field theory on the closed Euclidean 2-manifold $T_{\ell,\tau}$ (regarded as a bordism from $\emptyset$ to itself).

What can we say about the partition function Z_E? First of all, it is a *smooth* function. To see this, we note that the tori $T_{\ell,\tau}$ fit together to a smooth bundle

$$p \colon \Sigma \to \mathbb{R}_+ \times \mathbb{R}^2_+ \tag{61}$$

with a fiberwise Euclidean structure, such that the fiber over $(\ell, \tau) \in \mathbb{R}_+ \times \mathbb{R}^2_+$ is the Euclidean torus $T_{\ell,\tau}$. As discussed in the previous section, evaluating E on this smooth family results in a smooth function on the parameter space $\mathbb{R}_+ \times \mathbb{R}^2_+$. Compatibility of E with pullbacks guarantees that this is the function Z_E.

Secondly, the partition function Z_E is *invariant under the $SL_2(\mathbb{Z})$-action.* This follows by noting that the group $SL_2(\mathbb{Z})$ acts on the universal bundle (61) by bundle automorphisms which preserve the fiberwise Euclidean structure such that the projection map is equivariant. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ the associated bundle automorphism maps the fiber $T_{\ell,\tau}$ isometrically to the fiber $T_{\ell',\tau'}$ for $\ell' = \ell|c\tau + d|$, $\tau' = \frac{a\tau+b}{c\tau+d}$ via the map induced by the isometry $\mathbb{C} \to \mathbb{C}$ given by multiplication with the unit complex number $\frac{|c\tau+d|}{c\tau+d}$ (to check this claim, it is easier to show that multiplication by $\frac{c\tau+d}{|c\tau+d|}$ maps the lattice $\ell'(\mathbb{Z}\tau' + \mathbb{Z}1)$ to the lattice $\ell(\mathbb{Z}\tau + \mathbb{Z}1)$). Then Proposition 50 implies that the partition function Z_E is $SL_2(\mathbb{Z})$-equivariant.

Now we want to comment on how close Z_E comes to being a modular function, i.e., a function on $\mathbb{R}^2_+$ which is $SL_2(\mathbb{Z})$-invariant, holomorphic and holomorphic at infinity. Since the *conformal* class of the torus $T_{\ell,\tau}$ is independent of the scaling factor ℓ, the partition function of a *conformal* field theory E independent of ℓ. It follows that $Z_E(1,\tau)$ is a $SL_2(\mathbb{Z})$-invariant function on the upper half plane. If E is not conformal, there is no reason to expect $Z_E(\ell,\tau)$ to be independent of ℓ, and hence no reason for $Z_E(1,\tau)$ to be invariant under the $SL_2(\mathbb{Z})$-action. Similarly, even if E is a conformal theory, one shouldn't expect $Z_E(1,\tau)$ to be a *holomorphic* function, unless E is *holomorphic* in the sense that the operators associated to any bordism Σ depend holomorphically on the parameters determining the conformal structure on Σ. A precise definition of a holomorphic theory can be given in the terminology of this paper by working with families of conformal bordisms parametrized by complex analytic spaces instead of smooth manifolds.

Again, like in the cases $d = 0$ and $d = 1$ we've discussed before, Euclidean field theories of dimension 2 are disappointing from the point of view of a topologist who tries to find cocycles for elliptic cohomology. Again, supersymmetry comes to the rescue as shown by the following result.

Theorem 62. *Let $\widehat{E}$ be a Euclidean field theory of dimension $2|1$ and degree n. Then its partition function $Z_{\widehat{E}}\colon \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathbb{C}$ is independent of $\ell \in \mathbb{R}_+$, and as function of τ it is a weak integral modular form of weight $-\frac{n}{2}$.*

The partition function of a $2|1$-dimensional EFT $\widehat{E}$ is defined as the partition function of an associated non-supersymmetric EFT E of dimension 2. More precisely, E is a *spin* EFT of dimension d, i.e., a functor

$$E\colon d\text{-}\mathsf{EB}^{fam}_{spin} \longrightarrow \mathsf{TV}^{fam}_{super}$$

Here $\mathsf{TV}^{fam}_{super}$ is the internal category defined analogously to TV^{fam}, but using *super vector bundles* instead of vector bundles. The *spin bordism category* d-EB^{fam}_{spin} is defined exactly like d-EB^{fam} but the (fiberwise) Euclidean structure is augmented by the addition of a *spin structure*. This can be expressed by replacing the isometry group $\mathrm{Isom}(\mathbb{E}^d) = \mathbb{R}^d \rtimes SO(\mathbb{R}^d)$ by its spin double cover $\mathbb{R}^d \rtimes Spin(\mathbb{R}^d)$. This additional structure allows us to define a functor $\mathcal{S}$ of internal categories

$$\mathcal{S}\colon d\text{-}\mathsf{EB}^{fam}_{spin} \longrightarrow d|\delta\text{-}\mathsf{EB}^{fam},$$

where $d|\delta$-EB^{fam} is the restriction of $d|\delta$-EB^{sfam} to families paramatrized by ordinary manifolds, which makes $d|\delta$-EB^{fam} a category internal to $\mathsf{SymCat}/\mathsf{Man}$. The construction of $\mathcal{S}$ of course requires that it makes sense to talk about Euclidean structures on super manifolds of dimension $d|\delta$ as defined in section 4.2, i.e., that there is a complex spinor representation Δ of $Spin(d)$ of dimension δ, and a $Spin(d)$-equivariant, non-degenerate symmetric pairing $\Gamma\colon \Delta^*\otimes\Delta^*\to V_{\mathbb{C}}$. If Σ is a spin d-manifold with Euclidean structure, it has a principal $Spin(d)$-bundle $Spin(\Sigma)\to\Sigma$ determined by the spin structure. We obtain an associated complex spinor bundle $Spin(\Sigma)\times_{Spin(d)}\Delta^*\to\Sigma$ and the corresponding cs-manifold

$$\mathcal{S}(\Sigma) := \Pi\left(Spin(\Sigma)\times_{Spin(d)}\Delta^*\to\Sigma\right) \tag{63}$$

of dimension $d|\delta$, the *superfication* of Σ. The Euclidean charts of Σ (with transition functions in $\mathbb{R}^d\rtimes Spin(d)$) lead to super Euclidean charts of $\mathcal{S}(\Sigma)$ (with transition functions still in $\mathbb{R}^d\rtimes\mathrm{Spin}(d)$). In other words, the Euclidean structure on Σ induces a super Euclidean structure on $\mathcal{S}(\Sigma)$. Applying this construction to all (families of) Euclidean spin manifolds defines functors $\mathcal{S}$.

To construct E, we note that the category of manifolds Man is a subcategory of the category of super manifolds SMan. This allows us to restrict any category C fibered over SMan to a category $\mathsf{C}_{|\mathsf{Man}}$ fibered over Man. In particular, if $\mathsf{C}=(\mathsf{C}_0,\mathsf{C}_1)$ is a category internal to the 2-category $\mathsf{Cat}/\mathsf{SMan}$ of categories fibered over SMan, applying this to C_0, C_1 we obtain a category $\mathsf{C}_{|\mathsf{Man}}=((\mathsf{C}_0)_{|\mathsf{Man}},(\mathsf{C}_1)_{|\mathsf{Man}})$ internal to the 2-category $\mathsf{Cat}/\mathsf{Man}$. In addition, if C is internal to $\mathsf{SymCat}/\mathsf{SMan}$, the 2-category of symmetric monoidal categories fibered over SMan, then $\mathsf{C}_{|\mathsf{Man}}$ will be internal to $\mathsf{SymCat}/\mathsf{Man}$. If $f\colon\mathsf{C}\to\mathsf{D}$ is an internal functor of categories internal to $\mathsf{SymCat}/\mathsf{SMan}$, it restricts to an internal functor $f_{|\mathsf{Man}}$ of categories internal to $\mathsf{SymCat}/\mathsf{Man}$. In particular, $\widehat{E}$ restricts to an internal functor

$$d|\delta\text{-}\mathsf{EB}^{fam}\overset{\mathrm{def}}{=}d|\delta\text{-}\mathsf{EB}^{sfam}_{|\mathsf{Man}}\xrightarrow{\widehat{E}_{|\mathsf{Man}}}\mathsf{TV}^{sfam}_{|\mathsf{Man}}=\mathsf{TV}^{fam}_{super}.$$

Definition 64. Let E be a d-dimensional EFT. We say that E *extends to a $d|\delta$-dimensional EFT $\widehat{E}$* if E is equal to the composition

$$d\text{-}\mathsf{EB}^{fam}_{spin}\xrightarrow{\quad\mathcal{S}\quad}d|\delta\text{-}\mathsf{EB}^{fam}\xrightarrow{\quad\widehat{E}_{|\mathsf{Man}}\quad}\mathsf{TV}^{fam}_{super}$$

We now begin with the outline of the proof of Theorem 62, which consists of two quite separate parts. To explain these, we need a little bit of terminology first. Fix a real number $\ell>0$ and consider for $\tau\in\mathbb{R}^2_+$ the parallelogram in $\mathbb{R}^2$ spanned by the vectors ℓ and $\ell\tau$. The torus $T_{\ell,\tau}$ is obtained by gluing the opposite sides of this parallelogram; gluing only the two non-horizontal sides results in a cylinder $C_{\ell,\tau}$ with two boundary circles of length ℓ. Hence we can regard $C_{\ell,\tau}$ as a Euclidean bordism from the circle S^1_ℓ of length ℓ to itself (here we suppress the 2-dimensional Riemannian neighborhood of S^1_ℓ since this is simply a cartesian product of S^1_ℓ with an interval in the case at hand).

Let E be a 2-EFT, let $H := E_0(S^1_\ell)$ be the locally convex topological vector space associated to S^1_ℓ, and let

$$A\colon \mathbb{R}^2_+ \longrightarrow \mathcal{N}(H) \qquad \text{be defined by} \qquad A(\tau) = E_1(C_{\ell,\tau}). \tag{65}$$

Here $\mathcal{N}(H)$ is the algebra of trace class operators (also called nuclear operators) on H. We note that A is a *smooth* map, since the cylinders $C_{\ell,\tau}$ for ℓ fixed can be assembled in a smooth family of Euclidean bordisms parametrized by $\mathbb{R}^2_+$, to which E_1 associates a smooth family of trace class operators.

Proposition 66. *Let E be a spin EFT of dimension 2, let $H = E_0(S^1_\ell)$ be the associated $\mathbb{Z}/2$-graded topological vector space and $A\colon \mathbb{R}^2_+ \to \mathcal{N}(H)$ the associated smooth family of trace class operators. An extension of E to a supersymmetric EFT of dimension $2|1$ determines a smooth family $C\colon \mathbb{R}^2_+ \to \mathcal{N}(H)^{odd}$ of odd trace class operators commuting with the family A such that*

$$\partial_{\bar z} A = -C^2.$$

Proposition 67. *Let E be a spin EFT of dimension 2 and degree n which satisfies the conclusion of the previous proposition. Then the partition function of E is a weak integral modular form of weight $-\frac{n}{2}$.*

It is Proposition 66 that we consider as our original and new contribution, which shows in which way our *geometric* definition of supersymmetric field theories determines the algebra. In our previous paper [ST] we didn't yet have the proper formulation of the relevant geometric structure on super manifolds of dimension $2|1$. We speculated about the existence of such a structure in Hypothesis 3.29 of that paper, guided by the supersymmetry cancellation arguments which prove Proposition 67. These arguments seem to be fairly standard in the physics literature, at least on the Lie algebra level, i.e., for infinitesimal generators of the above families A, C. An outline can be found in our earlier survey paper [ST, Thm. 3.30].

Sketch of the proof of Proposition 66. The family $A(\tau)$ of trace class operators was obtained by applying E to the family of cylinders $C_{\ell,\tau}$ (for fixed $\ell \in \mathbb{R}_+$) parametrized by $\mathbb{R}^2_+$. If $\widehat{E}$ is the extension of E to a $2|1$-EFT, we will obtain the family of odd operators $B(\tau)$ by applying $\widehat{E}$ to a family of *super cylinders* $\Sigma \to \mathbb{R}^{2|1}_+$ parametrized by the super manifold $\mathbb{R}^{2|1}_+$. Here a super cylinder is a Euclidean super manifold of dimension $2|1$ whose reduced manifold is a Euclidean cylinder.

So far we've described families $\Sigma \to S$ by characterizing the fiber over every point $s \in S$. While this is intuitive, it is hardly satisfactory for ordinary bundles. If the base space is a super manifold, it certainly won't do, since talking about *points* of a super manifold can only reveal information about its reduced manifold. So a more global description of $\Sigma \to S$ is needed. It seems best to do that first in the non-super situation in a way that generalizes directly to the super case. We recall from Definition 40 that objects of 2-EB_0 are quadruples $Y = (Y, Y^c, Y^+, Y^-)$, where Y^c is a closed 1-manifold and Y is a 2-dimensional Euclidean neighborhood of Y^c, and $Y^\pm$ are the two pieces the complement $Y \smallsetminus Y^c$ consists of. In particular, to consider the circle S^1_ℓ as an object of 2-EB_0 we need to be precise about the Euclidean neighborhood. We define:

$$S^1_\ell = (\mathbb{R}^2/\ell\mathbb{Z}, \mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^2_+/\ell\mathbb{Z}, \mathbb{R}^2_-/\ell\mathbb{Z}).$$

Here $\mathbb{R} \subset \mathbb{R}^2$ is the x-axis, $\mathbb{R}^2_\pm \subset \mathbb{R}^2$ is the upper (resp. lower) half plane, and the group $\ell\mathbb{Z}$ acts on $\mathbb{R}^2$ via the emdeddings $\ell\mathbb{Z} \subset \mathbb{R} \subset \mathbb{R}^2$ and the translation action of $\mathbb{R}^2$ on itself. Below is a picture of the object S^1_ℓ.

Now we'll describe the cylinder $C_{\ell,\tau}$ as an object of 2-EB_1. We recall that an object of 2-EB_1 is a pair Y_0, Y_1 of objects of 2-EB_0 and bordism from Y_0 to Y_1, i.e., a triple

$$(W_1 \xrightarrow{i_1} \Sigma \xleftarrow{i_0} W_0)$$

where Σ is a Euclidean d-manifold, W_j is a neighborhood of $Y_j^c \subset Y_j$ for $j = 0, 1$, and i_0, i_1 are local isometries such that certain conditions are satisfied (see Definition 19 and picture 4). We make $C_{\ell,\tau}$ precise as an object of 2-EB_1 by declaring it to be the following bordism from S^1_ℓ to itself:

$$C_{\ell,\tau} = (\mathbb{R}^2/\ell\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{R}^2/\ell\mathbb{Z} \xleftarrow{\ell\tau} \mathbb{R}^2/\ell\mathbb{Z}),$$

where $\ell\tau\colon \mathbb{R}^2 \to \mathbb{R}^2$ is translation by $\ell\tau \neq 0$ and id is the identity, see figure 4.

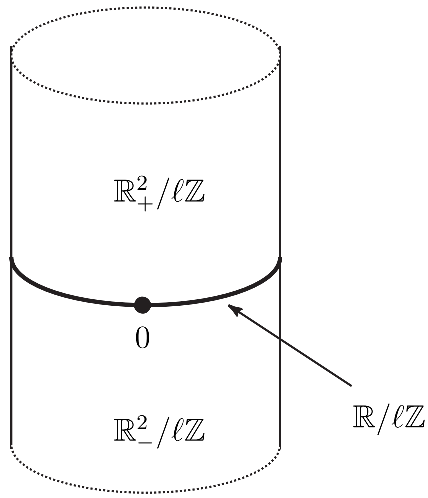

Figure 3: **The object** S^1_ℓ **of** 2-EB_0

Before moving on to the case of families and super manifolds, we make the following two simple geometric observations:

$$C_{\ell,\tau+1} = C_{\ell,\tau} \qquad C_{\ell,\tau} \circ C_{\ell,\tau'} \cong C_{\ell,\mu(\tau,\tau')}, \tag{68}$$

where $\mu\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is the addition map. We remark that the objects $C_{\ell,\tau+1}$ and $C_{\ell,\tau}$ are in fact the *same* object, since the map $\ell\tau\colon \mathbb{R}^2/\ell\mathbb{Z} \to \mathbb{R}^2/\ell\mathbb{Z}$ depends only on τ modulo $\mathbb{Z}$. The bordism $C_{\ell,\tau} \circ C_{\ell,\tau'}$ obtained by composing these cylinders is not equal to, but isomorphic to the cylinder $C_{\ell,\tau+\tau'}$ as object of 2-EB_1.

Now we turn to families. Suppose that S is a manifold and $\ell\colon S \to \mathbb{R}_+ \subset \mathbb{R}^2$, $\tau\colon S \to \mathbb{R}^2_+$ are smooth maps. We want to think of ℓ and τ as sections of the trivial bundle $S \times \mathbb{R}^2 \to S$. Since $\mathbb{R}^2$ is a group, left multiplication by a section can be interpreted as gauge transformation. Let

$$E = \ell\mathbb{Z}\backslash(S \times \mathbb{R}^2) \xrightarrow{p} S$$

be the bundle obtained by dividing out the free $\mathbb{Z}$-action given by multiplication by ℓ. The Euclidean structure on $\mathbb{R}^2$ induces a fiberwise Euclidean structure on E. Then we get an object $S^1_\ell := (S, Y, Y^c, Y^\pm)$ of 2-EB_0^{fam} by setting $Y := E$, $Y^c := \ell\mathbb{Z}\backslash(S \times \mathbb{R})$ and $Y^\pm := \ell\mathbb{Z}\backslash(S \times \mathbb{R}^2_\pm)$. Moreover, we obtain a bordism $C_{\ell,\tau} = (S, (W_1 \xrightarrow{i_1} \Sigma \xleftarrow{i_0} W_0)$ from S^1_ℓ to itself by setting

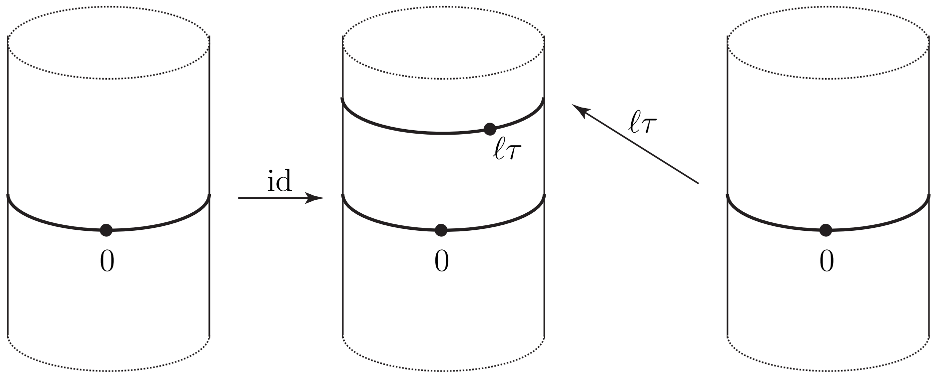

Figure 4: **The object $C_{\ell,\tau}$ of 2-EB_1**

$\Sigma = W_0 = W_1 = E$ and $i_1 = \mathrm{id}$. The map $i_0\colon E \to E$ is induced by left multiplication by the section $\ell\tau$. We note that if the parameter space S is a point, we obtain the objects S^1_ℓ, $C_{\ell,\tau}$ defined before.

Finally we come to the case of super manifolds. Let S be a supermanifold, and let $\ell\colon S \to \mathbb{R}_+ \subset \mathbb{R}^{2|1}$, $\tau\colon S \to \mathbb{R}^{2|1}_+$ be smooth functions. We make use of the fact that $\mathbb{R}^{2|1}$ is super Lie group (see **??**) and imitate the construction above by defining the bundle

$$E = \ell\mathbb{Z}\backslash(S \times \mathbb{R}^{2|1}) \xrightarrow{p} S.$$

Here some care is necessary since unlike $\mathbb{R}^2$, the super group $\mathbb{R}^{2|1}$ is *not commutative*. We remark that it is crucial that the $\mathbb{Z}$-action on $S \times \mathbb{R}^{2|1}$ is defined by *left* multiplication since only left multiplication leaves the super Euclidean structure on $\mathbb{R}^{2|1}$ invariant. This guarantees that E has a fiberwise super Euclidean structure. We define the objects

$$S^1_\ell \in 2|1\text{-}\mathsf{EB}_0^{sfam} \qquad \text{and} \qquad C_{\ell,\tau} \in 2|1\text{-}\mathsf{EB}_1^{sfam}$$

as before, observing that i_0 is well-defined since multiplication by $\ell\tau$ commutes with ℓ, despite the fact that $\mathbb{R}^{2|1}$ is non-commutative. We remark that the geometric relations (68) continue to hold in the category $2|1\text{-}\mathsf{EB}_1^{sfam}$.

Now let us apply the functor $\widehat{E}$ to derive algebraic consequences of these geometric relations. Applying $\widehat{E}_0\colon 2|1\text{-}\mathsf{EB}_0^{sfam} \to \mathsf{TV}^{sfam}$ to S^1_ℓ gives a vector bundle over S. Let us assume that $\ell\colon S \to \mathbb{R}_+$ is a *constant* map. Then this vector bundle is trivial with fiber $H = \widehat{E}_0(S^1_{\ell\colon \mathrm{pt}\to\mathbb{R}_+})$, due to compatibility with pullbacks. Applying $\widehat{E}_1\colon 2|1\text{-}\mathsf{EB}_1^{sfam} \to \mathsf{TV}_1^{sfam}$ to the bordism $C_{\ell,\tau}$ gives a vector bundle automorphism of $S \times H$ which we can reinterpret as a smooth map

$$\widehat{E}_1(C_{\ell,\tau})\colon S \longrightarrow \mathcal{N}(H)$$

to the space $\mathcal{N}(H)$ of nuclear (= trace class) operators on the locally convex vector space H. In particular, for $\tau = \mathrm{id}\colon \mathbb{R}_+^{2|1} \to \mathbb{R}_+^{2|1}$ we obtain a smooth function

$$\widehat{A} := \widehat{E}_1(C_{\ell,\mathrm{id}})\colon \mathbb{R}_+^{2|1} \longrightarrow \mathcal{N}(H).$$

We observe that the morphisms in the category TV_1^{sfam} are determined by their images under the source and target functors $s,t\colon \mathsf{TV}_1^{sfam} \to \mathsf{TV}_0^{sfam}$. This implies that the *isomorphism* in the relation (68) leads to the *identity*

$$E_1(C_{\ell,\tau}) \circ E_1(C_{\ell,\tau'}) = E_1(C_{\ell,\mu(\tau,\tau')}).$$

Specializing to the case where $S = \mathbb{R}_+^{2|1} \times \mathbb{R}_+^{2|1}$ and τ, τ' are the projection maps onto the first resp. second factor, we obtain the following identity.

$$\widehat{A}(z_1, \bar{z}_1, \theta_1) \circ \widehat{A}(z_2, \bar{z}_2, \theta_2) = \widehat{A}(z_1 + z_2, \bar{z}_1 + \bar{z}_2 + \theta_1\theta_2, \theta_1 + \theta_2) \tag{69}$$

Let us write the function $\widehat{A}$ in the form $\widehat{A} = A + \theta B$, where A (resp. B) is an even (resp. odd) element of $C^\infty(\mathbb{R}^{2|1}) \otimes \mathcal{N}(H)$. We claim that A is in fact the function defined earlier in (65). To see this we note that for any $\tau\colon S \to \mathbb{R}_+^2$ the restriction of the bordism family $C_{\ell,\tau}^{2|1}$ to $S_{red} \subset S$ agrees with the *superfication construction* $\mathcal{S}$ applied to the family $C_{\ell,\tau_{red}}^2$, where τ_{red} is the restriction of $\tau\colon S \to \mathbb{R}_+^2$ to S_{red}. Compatibility of $\widehat{E}$ with pullbacks then implies that the restriction of $\widehat{A}$ to $\mathbb{R}_+^2 = (\mathbb{R}_+^{2|1})_{red}$ is equal to A.

Writing $\widehat{A}$ in this form, we now expand both sides of the relation (69) as powers of θ_1 and θ_2. Let us consider only terms involving $\theta_1\theta_2$. On the left hand side we obtain

$$\theta_1 B(z_1) \circ \theta_2 B(z_2) = -\theta_1\theta_2 B(z_1) \circ B(z_2),$$

since θ_2 and $B(z_1)$ are both odd. On the right hand side the only term involving the product $\theta_1\theta_2$ comes from the Taylor expansion

$$A(z_1 + z_2, \bar{z}_1 + \bar{z}_2 + \theta_1\theta_2) = A(z_1 + z_2, \bar{z}_1 + \bar{z}_2) + \frac{\partial A}{\partial \bar{z}}(z_1 + z_2, \bar{z}_1 + \bar{z}_2)\theta_1\theta_2$$

Comparing coefficients we conclude

$$\frac{\partial A}{\partial \bar{z}}(z_1+z_2) = -B(z_1)\circ B(z_2).$$

In particular, setting $z_1 = z_2 = \frac{z}{2}$ and $C(z) := B(\frac{z}{2})$ we obtain a family with the desired properties, provided we can show that all operators of this family commute with all operators of the family A. We will again use equation (69) to prove this. Again expanding both sides in powers of θ_1 and θ_2, this time we look for the terms only involving θ_1.This leads to the identity

$$B(z_1)\circ A(z_2) = B(z_1+z_2).$$

Similarly, comparing the coefficients of θ_2 leads to

$$A(z_1)\circ B(z_2) = B(z_1+z_2),$$

which shows that the A-family and the B-family commute with each other.

□

Outline of the proof of Proposition 67. Let E be a 2-dimensional EFT. Then its partition function $Z_E(\ell,\tau)$ can be written as

$$Z_E(\ell,\tau) = E_1(T^{2|1}_{\ell,\tau}) = \operatorname{str} E_1(C^{2|1}_{\ell,\tau}) = \operatorname{str} A(\ell,\tau).$$

Now using the assumption of our Proposition, we write

$$\frac{\partial A}{\partial \bar{z}} = -C^2 = -\frac{1}{2}[C,C]$$

(where $[\quad,\quad]$ is the *graded* commutator) and hence :

$$\frac{\partial}{\partial \bar{z}}\operatorname{str} A = -\operatorname{str}(C^2) = -\frac{1}{2}\operatorname{str}[C,C] = 0,$$

since the super trace vanishes on graded commutators. This shows that $\operatorname{str} A$ is a *holomorphic* function on the upper half plane $\mathbb{R}^2_+$.

We observe that the relations (68) imply that the operator $A(\tau) = E_1(C_{\ell,\tau})$ depends only on $q = e^{2\pi i\tau}$ and that $\tau \mapsto A(\tau)$ is a commutative semi-group of trace class operators parametrized by the upper half-plane $\mathbb{R}^2_+$. This allows to use the spectral theorem to decompose H into a sum of simultaneous (generalized) eigenspaces for these operators. Each non-zero eigenvalue $\lambda(\tau)$ is then a smooth homomorphisms $\mathbb{R}^2_+ \to \mathbb{C}^*$ and hence can be written in the form

$$\lambda(\tau) = e^{2\pi i(a\tau - b\bar{\tau})} = q^a \bar{q}^b$$

for some $a, b \in \mathbb{C}$. We note that $A(\tau+1) = A(\tau)$ implies $\lambda(\tau+1) = \lambda(\tau)$ which in turn implies $a - b \in \mathbb{Z}$. Let us denote by $H_{a,b} \subset H$ the generalized eigenspace corresponding to the eigenvalue function $\lambda(\tau)$ given by equation (**??**). We note that the spaces $H_{a,b}$ are *finite dimensional*, since the operators $A(\tau)$ are trace class and hence compact; in particular, any generalized eigenspace with non-zero eigenvalue is finite dimensional.

Since only the non-zero eigenspaces contribute to the super trace of $A(\tau)$, we have

$$\operatorname{str} A(\tau) = \sum_{a,b} \operatorname{str}(A(\tau)_{|H_{a,b}}).$$

It is straightforward to calculate the super trace of $A(\tau)$ restricted to $H_{a,b}$; $A(\tau)$ is an even operator and hence it maps the even (resp. odd) part of H to itself, and we can calculate the trace of $A(\tau)$ acting on $H^{\pm}_{a,b}$ separately. There is a basis of $H^{\pm}_{a,b}$ such that the matrix corresponding to $A(\tau)$ is upper triangular with diagonal entries $\lambda_{a,b}(\tau)$. It follows that

$$\operatorname{str}\left(A(\tau)_{|H_{a,b}}\right) = \lambda_{a,b}(\tau) \operatorname{sdim} H_{a,b}.$$

We note that the argument proving the holomorphicity of $\operatorname{str} A(\tau)$ continues to hold if we restrict $A(\tau)$ to the subspace $H_{a,b}$ (the projection map onto $H_{a,b}$ is built by *functional calculus* from the operators $A(\tau)$; hence any operator that commutes with all $A(\tau)$'s – like $B(\tau/2)$ – will also commute with the projection operator and hence preserve the subspace $H_{a,b}$). We note that the function $\lambda_{a,b}(\tau)$ is holomorphic if and only if $b = 0$. It follows:

$$\operatorname{sdim} H_{a,b} = 0 \qquad \text{for} \quad b \neq 0.$$

In particular, the only contribution to the super trace of $A(\tau)$ comes from the space $H_{a,0}$, which forces a to be an integer k. We conclude

$$\operatorname{str} A(\tau) = \sum_{k\in\mathbb{Z}} \operatorname{str}\left(A(\tau)_{|H_{k,0}}\right) = \sum_{k\in\mathbb{Z}} \lambda_{k,0}(\tau) \operatorname{sdim} H_{k,0} = \sum_{k\in\mathbb{Z}} q^k \operatorname{sdim} H_{k,0}.$$

The eigenspaces $H_{k,0}$ must be trivial for sufficiently negative k, otherwise the corresponding eigenvalues q^k are arbitrarily large and $A(\tau)$ cannot be of trace-class. $\square$

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