PULLING APART 2-SPHERES IN 4–MANIFOLDS

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Abstract. An obstruction theory for representing homotopy classes of surfaces in 4–manifolds by immersions with pairwise disjoint images is developed using the theory of *non-repeated* Whitney towers. Generalizations and geometric proofs of some results of Milnor and of Casson are given...

1. Introduction

This paper applies the theory of Whitney towers, as introduced in [31], to address the problem of representing homotopy classes of 2–spheres in 4–manifolds by immersions with pairwise disjoint images. This problem has a well-known solution for a single pair of maps: Wall's intersection pairing λ takes values in the integral group ring $\mathbb{Z}[\pi]$ of the fundamental group $\pi := \pi_1 X$ of the ambient 4–manifold, and this intersection "number" $\lambda(A_1, A_2) \in \mathbb{Z}[\pi]$ is the complete obstruction to homotoping (rel boundary) a pair of (simply-connected) surfaces A_1 and A_2 to be disjoint. Wall's pairing associates a sign and an element of π to each transverse intersection point between the surfaces and the vanishing of $\lambda(A_1, A_2)$ implies that all of these intersections can be paired by *Whitney disks*. These Whitney disks can be used to pull apart A_1 and A_2 by first pushing any intersection points between A_2 and the interiors of the Whitney disks down into A_2, and then using the Whitney disks to guide Whitney moves on A_1 to eliminate all intersections between A_1 and A_2 (Figures 1 and 2). Note that in the presence of a third surface A_3 this

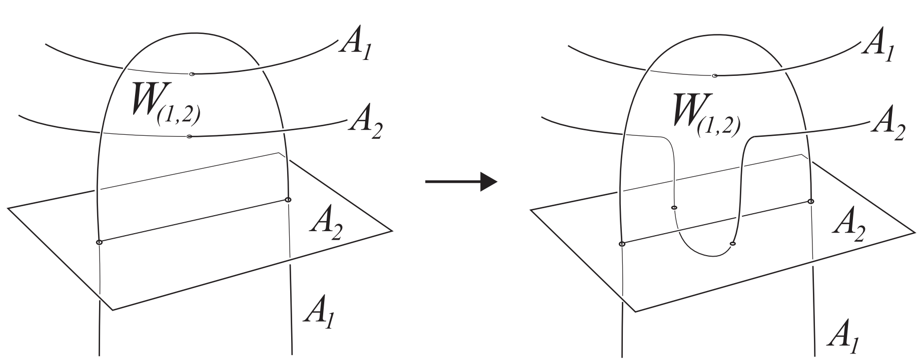

Figure 1. Pushing down an intersection between a sheet of the surface A_2 and the interior of a Whitney disk $W_{(1,2)}$.

procedure appears to break down because it is not clear how to eliminate an intersection

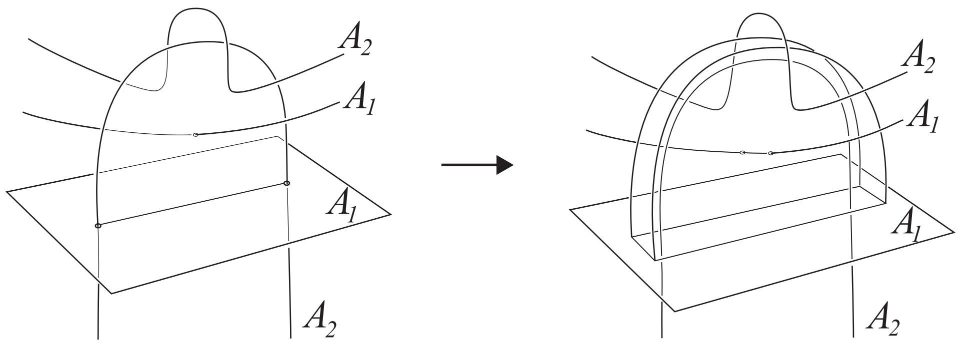

FIGURE 2. A Whitney move guided by the Whitney disk of Figure 1. The left hand side of this figure differs from the right hand side of Figure 1 only by a rotation of coordinates.

point between one surface and a Whitney disk that pairs intersections between the other two surfaces. In fact, such "higher order" intersections were used in [30] (and earlier for π trivial in [26, 37]) to define an invariant $\lambda(A_1, A_2, A_3)$ which takes values in a quotient of $\mathbb{Z}[\pi \times \pi]$ and is the complete obstruction to homotoping apart a triple A_1, A_2, A_3 with pairwise vanishing Wall invariants $\lambda(A_i, A_j)$. In this case, the procedure for separating the surfaces involves constructing "second order" Whitney disks which pair the intersections between surfaces and Whitney disks. The existence of these second order Whitney disks allows for an analogous pushing-down procedure which only creates self-intersections and cleans up the Whitney disks enough to pull apart the surfaces by an ambient homotopy.

Building on these ideas, we will describe an obstruction theory in terms of *non-repeated Whitney towers* and *non-repeated intersection trees* $\lambda_n(A_1, A_2, \ldots, A_m)$. The *order* of a non-repeated Whitney tower determines how many of the underlying surfaces at the bottom of the tower can be made pairwise disjoint by a homotopy. These non-repeated Whitney towers are special cases of the Whitney towers defined in [31] (see also [28, 29, 30, 32]). We sketch an introduction to these notions here and give more details in Section 2.

1.1. **Whitney towers.** Let $A_1, \ldots, A_m$ be properly immersed simply-connected surfaces in a 4–manifold X. Here *properly immersed* means that interior is generically immersed in interior and boundary is embedded in boundary. To begin our obstruction theory, we define the order of each A_i to be zero and say that the collection $\{A_i\}$ forms a *Whitney tower of order 0.* If the singularities among the A_i can be paired by Whitney disks then we get a Whitney tower of order 1 which is the union of these *order 1 Whitney disks* and the order 0 Whitney tower. If we only have Whitney disks pairing the intersections between distinct order 0 surfaces, then we get an order 1 *non-repeating* Whitney tower. Continuing on, an *order 2 Whitney tower* would also include Whitney disks (of *order 2*) pairing all the intersections between the 1st order Whitney disks and the order 0 surfaces.

An order 2 *non-repeated* Whitney tower would only require second order Whitney disks for intersections between an A_i and Whitney disks pairing A_j and A_k, where i, j and k are distinct. All of this generalizes to higher order, including the distinction between non-repeated and repeated intersection points, however things get more subtle as different types of intersections of the same order can appear.

If the A_i are homotopic (rel boundary) to pair-wise disjoint immersions, then we say that the A_i can be *pulled apart*. As a first step towards determining whether or not the A_i can be pulled apart, we have the following translation of the problem into the language of Whitney towers:

Proposition 1.1. *If $A_1, \ldots, A_m$ admit a non-repeated Whitney tower of order $m-1$, then the A_i can be pulled apart.*

The existence of a non-repeated Whitney tower of sufficient order encodes "pushing down" homotopies and Whitney moves which lead to disjointness as will be seen in the proof of Proposition 1.1 (Section 3). An immediate advantage of this point of view is that the higher order intersection theory of [31] can be applied inductively to increase the order of a Whitney tower or, in some cases, detect obstructions to doing so. The main idea is that to each unpaired intersection point p in an order n Whitney tower $\mathcal{W}$ one can associate a decorated unitrivalent tree t_p which bifurcates down from p through the Whitney disks to the order 0 surfaces A_i. The *order* of p is the number of trivalent vertices in t_p. The univalent vertices of t_p are labelled by the A_i (or just the indices i) and the edges of t_p are decorated with elements of the fundamental group $\pi := \pi_1 X$ of the ambient 4–manifold. Orientations of the A_i determine vertex-orientations and a sign $\epsilon_p \in \{\pm\}$ for t_p, and the *order n intersection tree* $\tau_n(\mathcal{W})$ is defined as the sum

$$\tau_n(\mathcal{W}) := \sum \epsilon_p \cdot t_p \in \mathcal{T}_n(\pi, m)$$

over all order n intersection points p in $\mathcal{W}$. Here $\mathcal{T}_n(\pi, m)$ is a free abelian group generated by order n decorated trees modulo relations which include the usual antisymmetry (AS) and Jacobi (IHX) relations of finite type theory Restricting to non-repeated intersection points in a non-repeated Whitney tower $\mathcal{W}$, yields the analogous order n *non-repeated intersection tree* $\lambda_n(\mathcal{W})$:

$$\lambda_n(\mathcal{W}) := \sum \epsilon_p \cdot t_p \in \Lambda_n(\pi, m)$$

which takes values in the subgroup $\Lambda_n(\pi, m) < \mathcal{T}_n(\pi, m)$ generated by trees whose univalent vertices have *distinct* labels.

In the case $n = 0$, our notation $\lambda_0(A_1, A_2, \ldots, A_m) \in \Lambda_0(\pi, m)$ just describes Wall's hermitian intersection pairing $\lambda(A_i, A_j) \in \mathbb{Z}[\pi]$ on the spheres $A_1, A_2, \ldots, A_m$.

For $n = 1$, we showed in [30] that if $\lambda_0(A_1, A_2, \ldots, A_m)$ vanishes, then taking

$$\lambda_1(A_1, A_2, \ldots, A_m) := \lambda_1(\mathcal{W})$$

in an appropriate quotient of $\Lambda_1(\pi, m)$ gives a homotopy invariant (for any choice of non-repeating Whitney tower $\mathcal{W}$ on the A_i).

In Theorem 2 of [31] it was shown that the vanishing of $\tau_n(\mathcal{W})$ implies that, after a homotopy, the A_i admit an order $n+1$ Whitney tower. The same arguments show that the vanishing of $\lambda_n(\mathcal{W})$ yields an order $n+1$ non-repeating tower. Thus, Proposition 1.1 implies the following theorem which was stated in [31]:

Theorem 1. *If $A_1, \dots, A_m$ admit a non-repeated Whitney tower $\mathcal{W}$ of order $(m-2)$ such that $\lambda_{m-2}(\mathcal{W})$ vanishes in $\Lambda_{m-2}(\pi, m)$, then the A_i can be pulled apart.*

1.2. **Pulling apart parallel surfaces.** The next theorem generalizes Milnor's surprising result that the components of any link of 0-parallel knots in the 3–sphere bound disjoint immersed disks into the 4–ball ([24]).

Theorem 2. *If A is an immersed 2-sphere in a simply connected 4-manifold with $[A] \cdot [A] = 0$, then any number of parallel copies of A can be pulled apart.*

Here $[A] \cdot [A] \in \mathbb{Z}$ is the usual homological self-intersection number of $[A] \in H_2(X; \mathbb{Z})$, and parallel copies of A are normal sections. Theorem 2 applies also to properly immersed disks, and our proof is completely geometric, in contrast to Milnor's algebraic proof of the above mentioned result in [24]. Note that Theorem 2 is *not* true in 4-manifolds with arbitrary fundamental group, as can be detected by the invariant $\tau(A)$ of [30] (see Section 4, which also contains the proof of Theorem 2).

1.3. **Relation to link-homotopy and Milnor's invariants.** A *link-homotopy* of an m-component link $L = L_1 \cup L_2 \cup \cdots \cup L_m$ in the 3-sphere is an ambient homotopy of L which preserves disjointness of the link components, i.e. during the homotopy only self-intersections of the L_i are allowed. In order to study "linking modulo knotting", Milnor ([24]) introduced the equivalence relation of link-homotopy (originally just called "homotopy") and defined his (non-repeating) μ-invariants, showing in particular that a link is link-homotopically trivial if and only if it has all vanishing μ-invariants. In the setting of link homotopy, Milnor's algebraically defined μ-invariants are intimately connected to non-repeating intersection trees as we describe next.

Let L be an m-component link in $S^3 = \partial B^4$ bounding m properly immersed 2–disks D_i into the 4-ball B^4. If the D_i admit an order n non-repeating Whitney tower $\mathcal{W}$, define $\lambda_n(L) := \lambda_n(\mathcal{W}) \in \Lambda_n(m)$.

Theorem 3.

(i) *$\lambda_n(L)$ is a link-homotopy invariant of L.*
(ii) *$\lambda_n(L)$ vanishes if and only if L bounds D_i admitting an order $n+1$ non-repeating Whitney tower.*

As a corollary, we see that an m-component link L is link-homotopically trivial if and only if $\lambda_n(L)$ vanishes for all $n = 0, 1, 2, \dots, m-1$. This follows from Theorem 1 together with the fact that L bounding disjointly immersed disks is equivalent to L being link-homotopically trivial ([11, 12]).

Thus, in the setting of link-homotopy, the non-repeating intersection tree carries the same information as the μ-invariants. Theorem 3 is proved by showing that the intersection tree computes the link longitudes in the Milnor group (see Section 5). This relationship suggests that Milnor invariants should generalize to intersection invariants of immersed surfaces in 4-manifolds.

Remark 1.2. A precise description of the relationship between general (repeating) Whitney towers on the D_i and Milnor's $\overline{\mu}$-invariants (with repeating indices [25]) for L is given in [32]. In particular, it is shown in [32] that the intersection tree $\tau_n(\mathcal{W})$ of such an order n Whitney tower determines the link longitudes in the lower central quotients of the link group via a natural map which turns unrooted trees into commutators by summing over all choices of roots. The existence of torsion in this general setting presents new subtleties and suggests the possibility that $\tau_n(\mathcal{W})$ detects new link invariants, which are higher order analogues of Arf invariants.

1.4. **Indeterminacies from lower order intersections.** As illustrated, for instance, by Theorem 3 above and by the order 1 invariants of [30, 26], there are settings in which the intersection tree only depends on the homotopy classes of the underlying order 0 surfaces A_i, and gives the complete obstruction to finding a higher order Whitney tower on the A_i. In general however, more relations are needed in the target space to account for indeterminacies in the choices of possible Whitney towers on given order 0 surfaces. In particular, for Whitney towers in a 4-manifold X with $\pi_2 X$ non-trivial, there are indeterminacies which correspond to tubing the interiors of Whitney disks into immersed 2-spheres. These INT *intersection indeterminacies* are, in principle, inductively manageable in the sense that they are determined by strictly lower order intersection trees on generators of $\pi_2 X$. However, as we describe in Section 7, if one wants the resulting target space to carry exactly the obstruction to the existence of a higher order tower then this target may not have a group structure.

It is interesting to note that the INT indeterminacies are in general different from $\overline{\mu}$-invariant indeterminacies in that they may involve intersections between 2-spheres *other* than the A_i.

1.4.1. ***Casson's separation lemma***. In the case of the first order intersection invariants λ_1 and τ_1, the INT relations in the target group of Y-trees correspond to order zero intersections between the A_i and other 2-spheres (or $\mathbb{R}P^2$s if $\pi_1 X$ contains 2-torsion) as described in [30].

In particular, the next theorem shows that the presence of 2-spheres which are (algebraically) dual to the A_i not only kills the first order invariant but all higher obstructions as well.

This recovers a result of Casson (proved algebraically in the simply-connected setting [2]) and Quinn (proved using transverse spheres [27, 8]):

Theorem 4. *If* $\lambda_0(A_1, A_2, \ldots, A_m) = 0$*, and there exist immersed 2–spheres* $B_1, \ldots, B_m$ *such that* $\lambda_0(A_i, B_j) = \delta_{ij} \in \mathbb{Z}[\pi]$ *for all* i*, then the* A_i *can be pulled apart.*

Recall that λ_0 describes Wall's intersection pairing in $\mathbb{Z}[\pi]$ (see subsection 2.6). Note that there are no restrictions on any intersections among the B_i.

1.5. **Homotopy invariance.** It is reasonable to conjecture that, modulo INT indeterminacies, the non-repeating intersection intersection tree only depends on the homotopy classes (rel ∂) of the underlying order zero surfaces A_i in an arbitrary 4-manifold, and thus gives a complete answer to the question of whether or not the A_i can be pulled apart:

Conjecture 5. *Simply connected order zero surfaces $A = A_1, \ldots, A_m$ in a 4-manifold X can be pulled apart if and only if $\lambda_n(A)$ vanishes in $\Lambda_n(\pi, m)/INT_n(A)$ for $n = 1, 2, \ldots, m-2$.*

In Section 7 we give an exact formulation of the INT indeterminacies for an order 2 non-repeating intersection tree in a simply-connected 4-manifold, and make precise the conjectured complete obstruction to pulling apart four 2-spheres.

2. Preliminaries

This section contains a summary of Whitney tower notions and notations as described in detail in [7, 28, 29, 30, 31, 32].

2.1. Whitney towers.

Definition 2.1.

- A *surface of order 0* in a 4–manifold X is a properly immersed surface (boundary embedded in the boundary of X and interior immersed in the interior of X). A *Whitney tower of order 0* in X is a collection of order 0 surfaces.
- The *order of a (transverse) intersection point* between a surface of order n_1 and a surface of order n_2 is $n_1 + n_2$.
- The *order of a Whitney disk* is $n + 1$ if it pairs intersection points of order n.
- For $n \geq 0$, a *Whitney tower of order* $n + 1$ is a Whitney tower $\mathcal{W}$ of order n together with Whitney disks pairing all order n intersection points of $\mathcal{W}$. These top order disks are allowed to intersect each other as well as lower order surfaces.

The Whitney disks in a Whitney tower are required to be *framed* ([10]) and have disjointly embedded boundaries. Intersections in surface interiors are assumed to be transverse. A Whitney tower is *oriented* if all its surfaces (order 0 surfaces *and* Whitney disks) are oriented. A *based* Whitney tower includes a chosen basepoint on each surface (including Whitney disks) together with a *whisker* (arc) for each surface connecting the chosen basepoints to the basepoint of X.

We will assume our Whitney towers are based and oriented, although basepoints and orientations will usually be suppressed from notation.

2.2. **Trees for Whitney disks and intersection points.** Our trees are unitrivalent, with univalent vertices labelled from an index set, usually $\{1, 2, \ldots, m\}$, and trivalent vertices cyclically oriented. A tree is *nonrepeating* if its univalent labels are distinct. Sometimes edges will be oriented and decorated with elements of $\pi_1 X$. A *root* of a tree is a chosen univalent vertex. The *order* of a tree is the number of trivalent vertices. We next give a brief explanation of the following diagram, which summarizes the tree-Whitney tower correspondence (See also Figure 3).

$$\begin{array}{ccc} \textit{paired} \text{ intersections} & \longrightarrow & \textit{rooted} \text{ trees} \\ \downarrow & & \downarrow \\ \textit{un}\text{-paired intersection} & \longrightarrow & \textit{un}\text{-rooted trees} \end{array}$$

We use bracketings of elements from the index set as subscripts to index surfaces in a Whitney tower $\mathcal{W}$, writing A_i for an order 0 surface (dropping the brackets around the singleton i), $W_{(i,j)}$ for a 1st order Whitney disk that pairs intersections between A_i and A_j, and $W_{((i,j),k)}$ for a second order Whitney disk pairing intersections between $W_{(i,j)}$ and A_k, etc.. When writing $W_{(I,J)}$ for a Whitney disk pairing intersections between W_I and W_J, the understanding is that if a bracket I is just a singleton (i) then the surface $W_I = W_{(i)}$ is just the order 0 surface A_i. In general, the order of W_I is equal to the order of (i.e. the number of trivalent vertices of) $t(W_I)$.

The upper horizontal arrow in the diagram is defined inductively: To a Whitney disk $W_{(I,J)}$ pairing intersections between W_I and W_J in a Whitney tower $\mathcal{W}$ is associated the rooted tree $t(W_{(I,J)}) := t(W_I) * t(W_J)$ (called the *rooted product* of $t(W_I)$ and $t(W_J)$) which is gotten by gluing the root vertices of $t(W_I)$ and $t(W_J)$ together to a single vertex and then sprouting a rooted edge from this new vertex. The tree $t(W_{(I,J)})$ sits as a subset of $\mathcal{W}$ with its root edge (including the edge's trivalent vertex) sitting in the interior of $W_{(I,J)}$ and its edges bifurcating down through lower order Whitney disks. Sheet-changing paths connecting basepoints of adjacent Whitney disks determine elements of $\pi_1 X$ which label the corresponding tree edges.

The left downward arrow in the diagram corresponds to transverse intersection, and the unrooted tree t_p associated to an intersection point $p \in W_{(I,J)} \cap W_{(K,L)}$ is the *inner product* $t(W_{(I,J)}) \cdot t(W_{(K,L)})$ which identifies the roots into a single (non-vertex) point. If X is not simply-connected, then the notation $t(W_{(I,J)}) \cdot_g t(W_{(K,L)})$ indicates that the edge containing the identified roots is labeled by the element $g \in \pi_1 X$ which is determined by a sheet-changing path through p joining the basepoints of $W_{(I,J)}$ and $W_{(K,L)}$.

Fixing conventions ([31]), orientations on the order zero surfaces in $\mathcal{W}$ induce vertex orientations on the above trees, modulo antisymmetry relations described below. Thinking of $t(W_I)$ as a subset of $\mathcal{W}$, it can be arranged that the trivalent orientations of $t(W_I)$ are induced by the orientations of the corresponding Whitney disks: Note that the pair of edges which pass from a trivalent vertex down into the lower order surfaces paired by a Whitney disk determine a "corner" of the Whitney disk which does not contain the other edge of the trivalent vertex. If this corner contains the *positive* intersection point paired

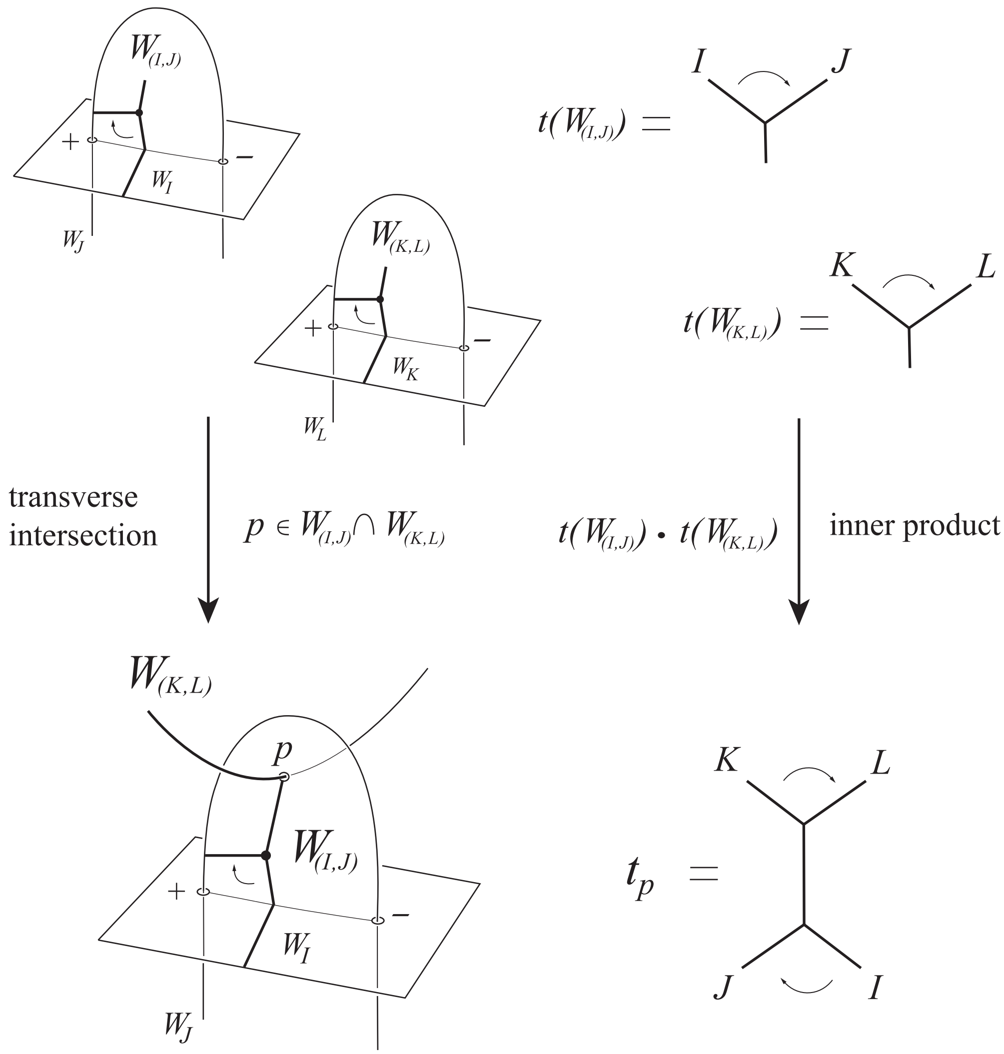

FIGURE 3

by the Whitney disk, then the vertex orientation and the Whitney disk orientation agree. Our figures are drawn to satisfy this convention.

2.3. **Nonrepeating Whitney towers.** Whitney disks and intersection points are called *nonrepeating* if their associated trees are nonrepeating (have distinct univalent labels), and *repeating* otherwise. An order n *nonrepeating Whitney tower* only has nonrepeating Whitney disks and has all nonrepeating intersections of order (strictly) less than n paired by (non-repeating) Whitney disks. In a non-repeating Whitney tower any repeating intersections are left unpaired.

2.4. **Intersection trees for Whitney towers.** For $\pi := \pi_1 X$, denote by $\mathcal{T}_n(m, \pi)$ the abelian group generated by order n (decorated) trees modulo relations illustrated in Figure 4. The edges are decorated by elements of π which are determined by sheet-

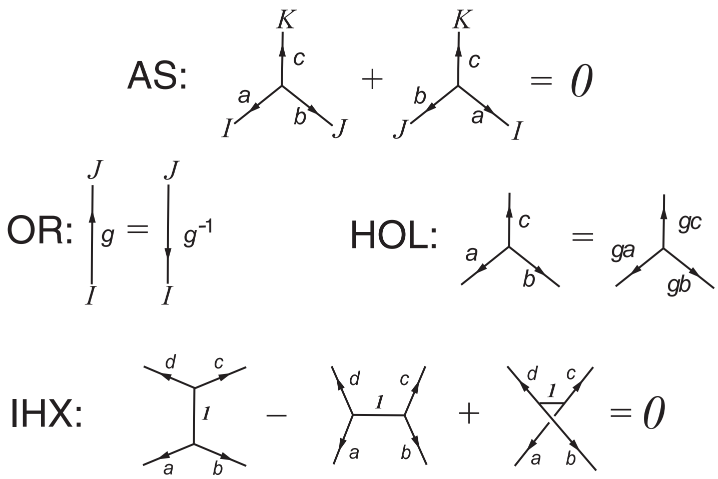

FIGURE 4

changing paths through basepoints in adjacent Whitney disks. Note that when X is simply connected, the edge decorations disappear and the relations reduce to the usual AS antisymmetry and IHX Jacobi relations. All the relations are homogeneous in the univalent labels, and restricting the generating trees to be nonrepeating order n trees defines the subgroup $\Lambda_n(m,\pi) < \mathcal{T}_n(m,\pi)$.

Definition 2.2. For an order n (oriented) Whitney tower $\mathcal{W}$ on $A_1,\ldots,A_m \looparrowright X$, the *order n intersection tree* $\tau_n(\mathcal{W})$ is defined by summing the signed trees $\pm t_p$ over all order n intersections $p \in \mathcal{W}$:

$$\tau_n(\mathcal{W}) := \sum \epsilon_p \cdot t_p \in \mathcal{T}_n(m,\pi).$$

Here the sign $\pm$ of t_p, for $p \in W_I \cap W_J$ is the usual sign of intersection between the oriented Whitney disks W_I and W_J.

If $\mathcal{W}$ is an order n nonrepeating Whitney tower, the order n *nonrepeating intersection intersection tree* $\lambda_n(\mathcal{W})$ is analogously defined by

$$\lambda_n(\mathcal{W}) := \sum \epsilon_p \cdot t_p \in \Lambda_n(m,\pi)$$

where the sum is over all order n nonrepeating intersections $p \in \mathcal{W}$.

2.5. **Order zero intersection trees.** The order zero intersection trees τ_0 and λ_0 correspond to Wall's hermitean intersection form μ and λ on a collection $A_1, A_2, \ldots, A_m$ of properly immersed surfaces in a 4-manifold X. The trees in $\tau_0(A_1, A_2, \ldots, A_m) \in \mathcal{T}_0(\pi, m)$ with both vertices labelled by the same index i correspond to Wall's self-intersection invariant $\mu(A_i)$, which is invariant under regular homotopy of A_i. For $\mu(A_i)$ to be a *homotopy* (not just regular homotopy) invariant, one must also mod out by a *framing relation* which kills trees labelled by the trivial element in π (see [32] for the higher order framing relations). Wall's homotopy invariant hermitean intersection pairing $\lambda(A_i, A_j) \in \mathbb{Z}[\pi]$ (for $i \neq j$) corresponds to the trees in $\lambda_0(A_1, A_2, \ldots, A_m) \in \Lambda_0(\pi, m)$ having one vertex labelled by i and the other vertex labelled by j, after applying the OR relation to orient the edges from i to j.

The $A_1, A_2, \ldots, A_m$ admit an order one Whitney tower if and only if $\tau_0(A_1, A_2, \ldots, A_m) = 0 \in \mathcal{T}_0(\pi, m)$, and admit an order one non-repeating Whitney tower if and only if $\lambda_0(A_1, A_2, \ldots, A_m) = 0 \in \Lambda_0(\pi, m)$.

2.6. **Order one intersection trees.** It was shown in [30], and for $\pi_1 X = 1$ and $m = 3$ in [26, 37], that for $A_1, A_2, \ldots, A_m$ admitting an order one Whitney tower (resp. nonrepeating Whitney tower) $\mathcal{W}$, the order one intersection tree $\tau_1(A_1, A_2, \ldots, A_m) := \tau_1(\mathcal{W})$ (resp. order one nonrepeating intersection tree $\lambda_1(A_1, A_2, \ldots, A_m) := \lambda_1(\mathcal{W})$) is a homotopy invariant of the $A_1, A_2, \ldots, A_m$, if taken in an appropriate quotient of $\mathcal{T}_1(\pi, m)$ (resp. $\Lambda_1(\pi, m)$) which is determined by order zero intersections between the A_i and immersed 2-spheres in X. These are the order one INT intersection relations which are described in [30] (in slightly different notation) and below in Section 7. For τ_1 there are also framing relations, but there are no framing relations for λ_n (for all n) because framings on Whitney disks can always be corrected by "boundary-twisting" which creates only repeating intersections.

Then $\tau_1(A_1, A_2, \ldots, A_m)$ (resp. $\lambda_1(A_1, A_2, \ldots, A_m)$) vanishes if and only if the A_i are homotopic to maps which admit an order 2 Whitney tower (resp. order 2 nonrepeating Whitney tower). In particular, $\lambda_1(A_1, A_2, A_3) \in \Lambda_1(\pi, 3)/INT$ is the complete obstruction to pulling apart three order zero surfaces.

2.7. **Order n intersection trees.** For $A_1, A_2, \ldots, A_m$ admitting a (nonrepeating) Whitney tower $\mathcal{W}$ of order n, if ($\lambda_n(\mathcal{W}) = 0 \in \Lambda_n(\pi, m)$) $\tau_n(\mathcal{W}) = 0 \in \mathcal{T}_n(\pi, m)$ then the A_i are homotopic (rel ∂) to A'_i admitting a (nonrepeating) Whitney tower of order $n + 1$. This was shown in [31] (Theorem 2) in the general repeating case, and the exact same arguments work restricting to the nonrepeating case. By geometrically realizing the relations in the target group of the intersection tree in a controlled manner, one can convert "algebraic cancellation" of trees to "geometric cancellation" of pairs of points (paired by next order Whitney disks).

3. Proof of Proposition 1.1

Proof. Let $\mathcal{W}$ be a non-repeated Whitney tower of order $m - 1$ on $A_1, A_2, \ldots, A_m$. If $\mathcal{W}$ contains no Whitney disks, then the A_i are pairwise disjoint. In case $\mathcal{W}$ does contain Whitney disks, we will describe how to use finger-moves and Whitney moves to eliminate the Whitney disks of $\mathcal{W}$ while preserving the (non-repeating) order $m - 1$. (Absolute value notation will be used to denote order.)

Consider a Whitney disk $W_{(I,J)}$ in $\mathcal{W}$ of maximal order ($1 \leq |W_{(I,J)}| \leq m - 1$). If $W_{(I,J)}$ is *clean* ($\operatorname{int} W_{(I,J)}$ contains no singularities) then do the $W_{(I,J)}$-Whitney move on either W_I or W_J. This eliminates $W_{(I,J)}$ (and the cancelling pair of intersections between

W_I and W_J) while creating no new intersections (of any kind), hence preserves the order of the resulting Whitney tower which we continue to denote by $\mathcal{W}$.

If any maximal order Whitney disk $W_{(I,J)}$ in $\mathcal{W}$ is not clean, then the singularities in the interior of $W_{(I,J)}$ are exactly a finite number of unpaired intersection points, all of which are repeating. (Since $W_{(I,J)}$ is of maximal order, the interior of $W_{(I,J)}$ contains no Whitney arcs; and since $|\mathcal{W}| = m-1$, any unpaired non-repeating intersections must be of order at least $m-1$, but such non-repeating intersections can only occur in a Whitney tower on more than m surfaces of order 0 (because trees of order $m-1$ have $m+1$ univalent vertices).) So, for any $p \in W_{(I,J)} \cap W_K$, at least one of (I,K) or (J,K) is a repeating bracket. Assuming that (I,K), say, is repeating, push p off of $W_{(I,J)}$ down into W_I by a finger move (Figure 5). This creates only a pair of repeating intersections between W_I and W_K. After pushing down all intersections in the interior of $W_{(I,J)}$ by finger-moves in this way, do the clean $W_{(I,J)}$-Whitney move on either W_I or W_J. Repeating this procedure on all maximal order Whitney disks eventually yields the desired order $m-1$ non-repeating Whitney tower on the A_i with no Whitney disks. □

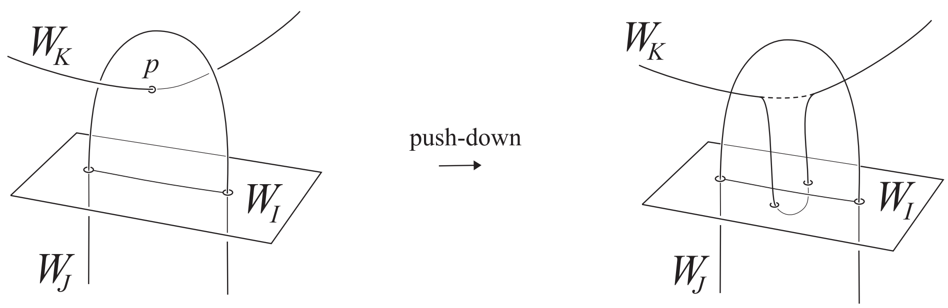

FIGURE 5

4. Pulling apart parallel 2-spheres

In this section we prove Theorem 2 of the Introduction. The proof includes a geometric proof that boundary links in the 3-sphere are link-homotopically trivial (Proposition 4.1 below). We also give an example illustrating that Theorem 2 is not true for *non*-simply connected 4-manifolds.

4.1. **Proof of Theorem 2.** Since $[A] \cdot [A] = 0$ and X is simply-connected, A supports an order one Whitney tower $\mathcal{W}$.

First consider the case where $\tau_1(A) := \tau(\mathcal{W}) = 0 \in \mathcal{T}_1(1) \cong \mathbb{Z}_2$ (by [26] this is equivalent to A having vanishing Arf invariant). In this case, A admits an order 2 Whitney tower, so by Lemma 3 of [29] for *any* $m \in \{3, 4, 5, \ldots\}$, A admits a Whitney tower $\mathcal{W}$ of order m. (The fact that A is *connected* and X is *simply connected* is crucial here, since under these hypotheses Lemma 3 of [29] shows that higher order Whitney towers can be built using a boundary-twisting construction.) Now, taking parallel copies of the Whitney disks in $\mathcal{W}$ yields an order m Whitney tower on $m+1$ parallel copies of A. In particular, we get an order m non-repeated Whitney tower (by throwing away the repeating Whitney disks) so, by Proposition 1.1, the $m+1$ parallel copies of A can be pulled apart.

This also proves the theorem in the case where X is closed: Then A is an integral multiple $d \cdot P$ of some immersed 2-sphere P representing a primitive class in $H_2(X) \cong \pi_2(X)$. Then $\tau_1(P)$ vanishes by the INT relation since P has a dual 2-sphere. Thus, any number of parallel copies of P may be pulled apart, which implies that any number of parallel copies of $A = d \cdot P$ can also be pulled apart.

Consider now the case where $\tau_1(A) = \tau_1(\mathcal{W})$ is the non-trivial element in $\mathbb{Z}_2$. We will first isolate (to a neighborhood of a point) the obstruction to building an arbitrarily high order Whitney tower, and then combine the previous argument away from this point with an application of Milnor's Theorem (which we will also prove geometrically in Proposition 4.1 below).

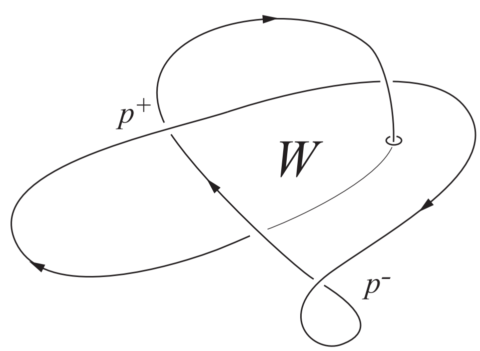

FIGURE 6

As illustrated in Figure 6, a trefoil knot in the 3-sphere bounds an immersed 2-disk in the 4-ball which supports an order 1 Whitney tower containing exactly one Whitney disk whose interior contains a single order one intersection point. It follows that the square knot, which is the connected sum of a right- and a left-handed trefoil knot, bounds an immersed disk D in the 4-ball which supports a Whitney tower containing exactly two first order Whitney disks, each of which contains a single order one intersection point. Being a well-known slice knot, the square knot also bounds an embedded 2-disk D' in the 4-ball, and by gluing together two 4-balls along their boundary 3-spheres we get an immersed 2-sphere $S = D \cup D'$ in the 4-sphere having the square knot as an "equator" and supporting the obvious order one Whitney tower $\mathcal{W}_S$ whose Whitney disks lie on D.

Now take $\mathcal{W}_S$ in a (small) 4-ball neighborhood of a point in X, and tube (connected sum) A into S. This does not change the (regular) homotopy class of A (so we will still denote this sum by A). Note that by construction there is a (smaller) 4-ball B^4 such that the intersection of the boundary ∂B^4 of B^4 with A is a trefoil knot, and B^4 contains one of the two Whitney disks of $\mathcal{W}_S$. Denote by A_0 the intersection of A with $X_0 := X \setminus B^4$.

Since the order one intersection point in the Whitney disk of $\mathcal{W}_S$ which is *not* contained in B^4 cancels the obstruction $\tau_1(\mathcal{W}) \in \mathbb{Z}_2$, we have that A_0 admits an order 2 Whitney tower in X_0, and hence again by Lemma 3 of [29], A_0 admits a Whitney tower of any order in X_0. As before, it follows that parallel copies of A_0 can be pulled apart by using parallel (non-repeating) copies of the Whitney disks in a high order Whitney tower on A_0. The parallel copies of A_0 restrict on their boundaries to a link of 0-parallel trefoil knots in ∂B^4, and the proof of Theorem 2 is completed by the following lemma which implies that these trefoil knots bound disjointly immersed 2-disks in B^4.

Proposition 4.1. *If the components L_i of a link $L = \cup L_i \subset S^3$ are the boundaries of disjointly embedded surfaces $F_i \subset S^3$ in the 3-sphere, then the L_i bound disjointly immersed 2-disks in the 4-ball.*

This proposition is a special case of the general results of [34] which are proved using symmetric surgery.

Proof. Choose a full symplectic basis of simple closed curves on each F_i bounding generically immersed 2-disks into the 4-ball. Note the interiors of these immersed disks are disjoint from $\cup_i F_i \subset S^3$, but may intersect each other. The proof proceeds inductively by using half of these disks to surger each F_i to an immersed disk F_i^0, while using the other half of the disks to construct Whitney disks which guide Whitney moves to recover disjointness.

We start with F_1. Let D_{1r} and D^*_{1r} denote the properly immersed disks bounded by the symplectic circles in F_1, with ∂D_{1r} geometrically dual to ∂D^*_{1r} in F_1.

Step 1: Using finger-moves, remove any interior intersections between the D^*_{1r} and any D_{1s} by pushing the D_{1s} down into F_1 (Figure 7).

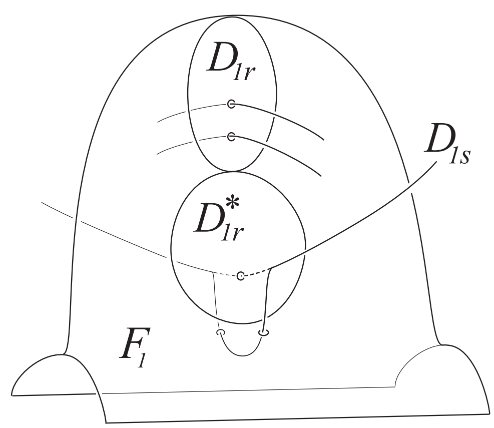

Figure 7

Step 2: Surger F_1 along the D_{1r} (Figure 8). The result is a properly immersed 2-disk F_1^0 in the 4-ball bounded by L_1 in S^3. The self-intersections in F_1^0 come from intersections and self-intersections in the surgery disks D_{1r}, and any intersections between the D_{1r} and

F_1 created in Step 1, as well as any intersections created by taking parallel copies of the D_{1r} during surgery. We don't care about any of these self-intersections in F_1^0, but we do want to eliminate all intersections between F_1^0 and any of the disks D_j on the other F_j, $j \geq 2$. These intersections between F_1^0 and the disks on the other F_j all occur in cancelling pairs, with each such pair coming from an intersection between a D_{1r} and a D_j. Each of these cancelling pairs admits a Whitney disk W_{1r}^* constructed by adding a thin band to (a parallel copy of) the dual disk D_{1r}^* as illustrated in Figure 8. Note that by Step 1 the interiors of the D_{1r}^* are disjoint from F_1^0, hence the interiors of the W_{1r}^* are also disjoint from F_1^0. The interiors of the W_{1r}^* may intersect the D_j, but we don't care about these intersections.

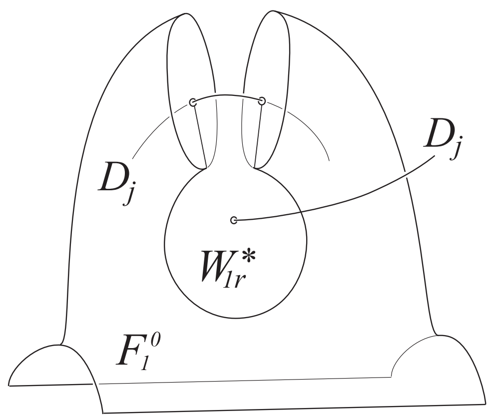

FIGURE 8

STEP 3: Do the W_{1r}^* Whitney moves on the D_j. This eliminates all intersections between F_1^0 and the disks D_j on all the other F_j ($j \geq 2$). Note that any interior intersections the W_{1r}^* may have had with the D_j only lead to more intersections among the D_j, so these three steps may be iterated, starting next by applying Step 1 to F_2. □

4.2. **Example.** If $\pi_1 X$ is non-trivial, then the conclusion of Theorem 2 may not hold, as we now illustrate. Let X be a 4-manifold with $\pi_1 X \cong \mathbb{Z}$, and let A be an immersed 0-framed 2-sphere admitting an order 1 Whitney tower $\mathcal{W}$ in X with a single order 1 intersection point p such that $\tau_1(A) = t_p \in \mathcal{T}_1(\mathbb{Z})$ is represented by the single Y-tree $Y(e, g, h) = t_p$ having one edge labelled by the trivial group element e, and the other edges labelled by non-trivial elements $g \neq h$, all edges oriented towards the trivalent vertex. Assume also that $\pi_2 X$ has trivial order 0 intersection form. Such examples are given in [30], and can be easily constructed by banding together the Borromean rings in the boundary of $B^3 \times S^1$ and attaching a 0-framed two handle.

If A_2 and A_3 are parallel copies of $A = A_1$, then the order 1 non-repeating intersection tree $\lambda_1(A_1, A_2, A_3)$ takes values in $\Lambda_1(\mathbb{Z}, 3)$ (since the vanishing of the first order intersections means that all INT_1 relations are trivial). By normalizing the label on the 3-labelled edge to the trivial element, using the HOL relations, $\Lambda_1(\mathbb{Z}, 3)$ is isomorphic to $\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}]$. Using six parallel copies of the Whitney disk in $\mathcal{W}$, we can compute that $\lambda_1(A_1, A_2, A_3)$ is represented by the sum of six Y-trees $Y(e, g, h)$, where the univalent vertex labels vary over the permutations of $\{1, 2, 3\}$. This element corresponds to the element

$$(g, h) - (h, g) + (hg^{-1}, g^{-1}) - (g^{-1}, hg^{-1}) + (gh^{-1}, h^{-1}) - (h^{-1}, gh^{-1}) \in \mathbb{Z}[\mathbb{Z} \times \mathbb{Z}]$$

which is non-zero if (and only if) g and h are distinct non-trivial elements of $\mathbb{Z}$.

5. Proof of Theorem 3

We will prove statement (i) of Theorem 3 by identifying the nth order non-repeating intersection tree $\lambda_n(L)$ with Milnor's link-homotopy μ-invariants of length $n{+}2$. Although this identification can be deduced from our description of intersection trees in terms of the Kontsevich integral [31], together with Habegger and Masbaum's paper [14], we prefer to describe here a direct connection between the combinatorially constructed Whitney towers and the algebraically defined Milnor invariants. The essential idea is that an nth order non-repeating Whitney tower can be used to compute the link longitudes as commutators in Milnor's nilpotent quotients of the fundamental group of the link complement.

Statement (i) then implies statement (ii) as follows: If $\lambda_n(L)$ vanishes, then (after a homotopy) the D_i admit an order $n+1$ non-repeating Whitney tower as described in Theorem 2 of [31]. On the other hand, if L bounds an order $n+1$ non-repeating Whitney tower $\mathcal{W}$, and L also bounds an order n non-repeating Whitney tower $\mathcal{W}'$ with $\lambda_n(\mathcal{W}') = \psi \neq 0 \in \Lambda_n(m)$, then gluing $\mathcal{W}$ and $\mathcal{W}'$ together along $L \subset S^3$ gives an order n non-repeating Whitney tower $\mathcal{W}''$ on immersed 2-spheres in S^4 with $\lambda_n(\mathcal{W}'') = \psi$. This contradicts statement (i), since we can then create a non-repeating Whitney tower bounded by the unlink U such that $\lambda_n(U) = \psi \neq 0$ by connect summing $\mathcal{W}'' \subset S^4$ into B^4 and tubing the 2-spheres into trivial disks bounded by the components of U.

The literature on Milnor's link invariants contains widely varying notations and conventions. We adopt basic notation from Habegger's papers (e.g. [14, 13]), except that the natural map from trees to commutators will be denoted η following Levine ([21, 22, 23]) (rather than 'j' as in [14]).

5.1. **Milnor's μ-invariants.** For a group G normally generated by $g_1, g_2, \ldots, g_m$, the *Milnor group* of G (with respect to the generating set) is the quotient of G by the subgroup $\ll [g_i, g_i^h] \gg$, $1 \leq i \leq m$, $h \in G$.

Let $L = L_1 \cup L_2 \cup \cdots \cup L_m \subset S^3$ be an m-component link. The *Milnor group* $\mathcal{M}(L)$ of L is the Milnor group of $\pi_1(S^3 \setminus L)$ with respect to a generating set of meridional elements. Specifically, $\mathcal{M}(L)$ has a presentation

$$\mathcal{M}(L) = \langle x_1, x_2, \ldots, x_m\,;\, [w_i, x_i],\, [x_j, x_j^g] \rangle$$

where the w_i are words in the meridional generators x_i determined by the link longitudes, and the g vary over all words in the generators, with $i, j \in \{1, 2, \ldots, m\}$. The Milnor group $\mathcal{M}(L)$ is the largest common quotient of the fundamental groups of all links which are link homotopic to L. See any of [1, 13, 14, 24] for details. Since $\mathcal{M}(L)$ only depends on the conjugacy classes of the meridional generators x_i, we can afford to (and will) suppress basepoints from all notation.

A presentation for the Milnor group of the unlink (or any link-homotopically trivial link) corresponds to the case where all $w_i = 1$, and Milnor's μ-invariants (with non-repeating indices) compare $\mathcal{M}(L)$ with this *free Milnor group* $\mathcal{FM}(m)$ by examining each longitudinal element in terms of the generators corresponding to the *other* components. Specifically, there is a canonical isomorphism between the direct sum of the lower central quotients $\oplus_{n\geq 1}\mathcal{FM}(m)_{(n)}/\mathcal{FM}(m)_{(n+1)}$ and $R\mathrm{Lie}(m) = \oplus_{n\geq 1}R\mathrm{Lie}_n(m)$, where the *reduced* free Lie algebra $R\mathrm{Lie}(m)$ is the quotient of the free Lie algebra on the x_i by the relations which set all repeating commutators equal to zero. Let $\mathcal{M}^i(L)$ denote the quotient of the Milnor group by the relation that sets $x_i = 1$. If the element in $\mathcal{M}^i(L)$ determined by the i-th longitude L_i lies in the n-th lower central subgroup $\mathcal{M}^i(L)_{(n)}$ for each i, then we have isomorphisms

$$\mathcal{M}(L)_{(n)}/\mathcal{M}(L)_{(n+1)} \cong \mathcal{FM}(m)_{(n)}/\mathcal{FM}(m)_{(n+1)} \cong R\mathrm{Lie}_n(m)$$

and the length n commutators $\mu^i_n(L) \in R\mathrm{Lie}^i_n(m)$ determined by the longitudes L_i are the *degree* n (also called *length* $n+1$) *μ-invariants* of L, where $R\mathrm{Lie}^i_n(m)$ is the reduced free Lie algebra on the $m-1$ generators x_j, for $j \neq i$.

5.2. **The maps η^i from trees to commutators.** For each i, define a map η^i from $\Lambda_n(m)$ to $R\mathrm{Lie}^i_{(n+1)}(m)$ by sending each tree t which has an i-labelled univalent vertex v_i to the commutator determined by t with a root at v_i. Trees without an i-labelled vertex are sent to zero. For example, if t is an order one Y-tree with univalent labels $1, 2, 3$, and cyclic vertex orientation $(1, 2, 3)$, then $\eta^1(t)$ is the commutator $[x_2, x_3]$, and $\eta^3(t) = [x_1, x_2]$, and $\eta^2(t) = [x_3, x_1]$. To be compatible with previous conventions for Whitney disk and grope orientations ([7, 5, 6, 31]), we use the commutator convention $[x, y] := x^{-1}y^{-1}xy$, for group elements x and y.

The maps η^i are injective since putting an i-label in place of the root in a tree corresponding to a commutator in $R\mathrm{Lie}^i_{(n+1)}(m)$ gives an inverse.

The maps η^i_n correspond to geometric constructions which desingularize an order n Whitney tower to a collection of embedded class $n+1$ *gropes*, as described in detail in [28]. Gropes are 2-complexes built by gluing together compact orientable surfaces, and this correspondence will be used in arguments below.

We will show that if $L \subset S^3$ bounds an order n non-repeating Whitney tower $\mathcal{W}$ on immersed disks $D_i \looparrowright B^4$, then for each i the longitude L_i lies in $\mathcal{M}^i(L)_{(n+1)}$, and $\eta^i_n(\lambda_n(L)) = \mu^i_{(n+1)}(L)$. Since the maps η^i are injective this will prove statement (i) of the theorem.

5.3. **Computing the μ-invariants from the intersection tree.** First consider the case where $L \subset S^3$ is an $(n+2)$-component link bounding an order n Whitney tower $\mathcal{W}_p \subset B^4$ which is a *split sub-tower*. This means that

- the components of L are the boundaries of the order zero properly embedded 2-disks $D_1, D_2, \ldots, D_{n+2} \hookrightarrow B^4$,
- $\mathcal{W}_p$ has a single unpaired intersection point p with associated tree t_p,
- the interior of each Whitney disk contains either p or a single boundary arc of a higher order Whitney disk, and no other singularities.

Now by Theorem 5 of [28], for any $i \in \{1, 2, \ldots, n+2\}$ we can use 0-surgeries and Whitney moves to "resolve" $\mathcal{W}_p$ so that L_i bounds an embedded class $n+1$ capped grope G_i^c in $B^4 \setminus \cup_{j\neq i} D_j'$, where the D_j' are pairwise disjointly embedded 2-disks homotopic (rel ∂) to the original D_j, and each cap of G_i^c is a meridional disk to a D_j'. Furthermore, the rooted tree $t(G_i^c)$ associated to G_i^c is just t_p with a root at the i-labelled univalent vertex. Since the D_j' are disjointly embedded, the inclusion $S^3 \setminus \cup_{j\neq i} L_j \hookrightarrow B^4 \setminus \cup_{j\neq i} D_j'$ induces isomorphisms on Milnor groups, with respect to a set of meridional generators x_j. Thus, G_i^c displays L_i as a single length $n+1$ commutator in the x_j, determining an element of $R\mathrm{Lie}_n^i(n+1)$ which we claim is $\eta^i(t_p)$. Since [28] deals with unoriented trees, the only thing that needs to be checked is that the sign works out. The invariance of oriented trees under the Whitney moves used to construct G_i^c is demonstrated in Lemma 14, Figures 10 and 11 of [31]. A typical 0-surgery which converts a Whitney disk $W_{(I,J)}$ into a cap $c_{(I,J)}$ is illustrated below in Figure 9, which also shows how the signed tree is preserved. The sign associated to the capped grope is the product of the signs coming from the intersections of the caps with the bottom stages which corresponds to the sign of the un-paired intersection point in the Whitney tower; (surgering along the other boundary arc of the Whitney disk, and the other sign cases are checked similarly). As described in [28], if either of the J- and K-labelled sheets is a Whitney disk, then the corresponding cap will be surgered after a Whitney move which turns the single cap-Whitney disk intersection into a cancelling pair of intersections between the cap and a Whitney disk.

Now consider any m-component link $L \subset S^3$ bounding an order n non-repeating Whitney tower $\mathcal{W}$ on immersed disks $D_i \looparrowright B^4$. As described in [28, 31], $\mathcal{W}$ can be *split*, so that all the Whitney disks of $\mathcal{W}$ are contained in split sub-towers $\mathcal{W}_{p_r}$ which are contained in pairwise disjointly embedded 4-balls B_r. Each B_r is a thickening of the tree t_{p_r} associated to $\mathcal{W}_{p_r}$, and $\lambda_n(\mathcal{W}) = \sum_r \lambda_n(\mathcal{W}_{p_r}) = \sum_r t_{p_r}$ (see Figure 10). Denoting by B^0 the punctured 4-ball which is the closure of the complement of the union of the B_r, the intersection $B^0 \cap (\cup_r B_r)$ is a collection of disjoint 3-spheres. The intersection of $\mathcal{W}$ with B^0 is a collection D^0 of pairwise disjointly immersed planar surfaces D_i^0. Each D_i^0 has L_i as one of its boundary components, and any other boundary components L_{r_i} of D_i^0 are boundaries of order zero disks D_{r_i} in distinct $\mathcal{W}_{p_r}$ which intersect D_i.

After performing finger-moves among the D_i^0 (which kill commutators between conjugate meridians and create only self-intersections), we can assume that the inclusion $S^3 \setminus L \hookrightarrow B^0 \setminus D^0$ induces isomorphisms on the corresponding Milnor groups. For each i,

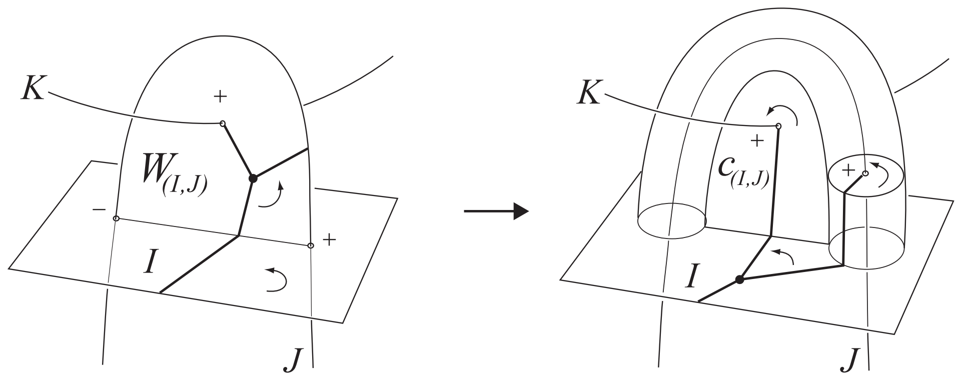

FIGURE 9. Resolving a Whitney tower to a capped grope preserves the associated oriented trees. The boundary of the I-labelled sheet represents the commutator $[x_J, x_K] := x_J^{-1}x_K^{-1}x_Jx_K$, up to conjugation, of the meridians x_J and x_K to the J- and K-labelled sheets.

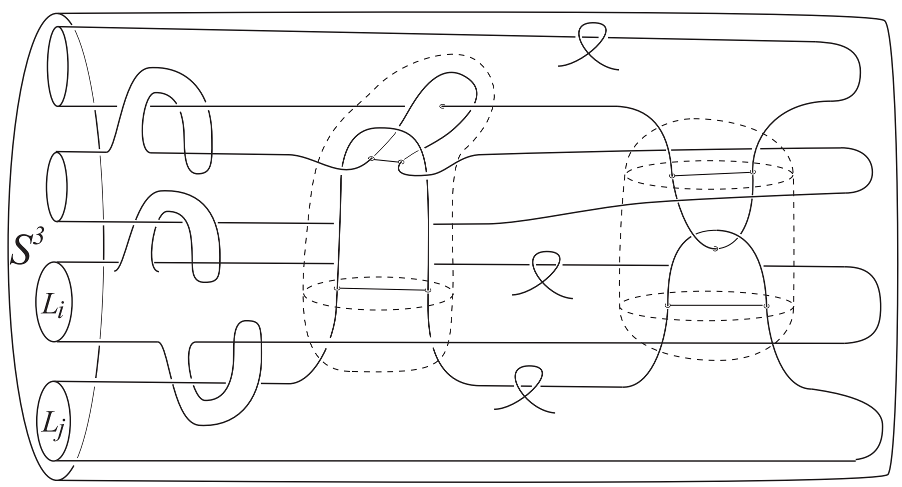

FIGURE 10. The 4-ball neighborhoods B_r of the split sub-towers are indicated by the dotted lines.

the planar surface D_i^0 displays L_i as a product of $L_{r_i} = \partial D_{r_i}$ up to conjugation. By Van Kampen's theorem and the above special case, we have that $\eta_n^i(\lambda_n(L)) = \sum \eta^i(t_{p_r}) = \mu^i_{(n+1)}(L)$ as desired.

6. Proof of Theorem 4

Proof. The pairwise vanishing of Wall's invariant gives an order 1 non-repeating Whitney tower. Assuming inductively for $1 \leq n < m-2$ the existence of an order n non-repeating Whitney tower $\mathcal{W}$ on the A_i, it is enough to show that it can be arranged that

$\lambda_n(\mathcal{W}) = 0 \in \Lambda_n(\pi, m)$, which allows us to find an order $n+1$ non-repeated Whitney tower (by Theorem 2 of [31]) and then to apply Theorem 1 when $n = m-2$.

By doing finger moves which realize the rooted product, any order n $\mathcal{W}$ can be modified (in a neighborhood of a 1-complex) to have an additional clean order n Whitney disk W_J whose decorated tree corresponds to any pre-chosen bracket J. If J is non-repeating and does not contain the label i, then tubing B_i into W_J will change $\lambda_n(\mathcal{W})$ exactly by $\pm\lambda_0(A_i, B_i) = \pm 1$ times the order n generator $t(J) \cdot_g t(i)$, where the sign and the element $g \in \pi$ can be chosen (arbitrarily) by the choice of tubing. For $1 \leq n$, any generator of $\Lambda_n(\pi, m)$ can be realized as t_p where p is an intersection point between an order 0 surface and a Whitney disk of order n, so the just described tubing procedure can be used to modify $\mathcal{W}$ until $\lambda_n(\mathcal{W}) = 0 \in \Lambda_n(\pi, m)$. □

7. Order 2 intersection indeterminacies in a simply-connected 4-manifold

The goal of this section is to give a precise formulation of the INT intersection indeterminacy relations which we conjecture to be necessary and sufficient to describe the target for the complete obstruction to pulling apart four simply-connected surfaces in a simply-connected 4-manifold X:

Conjecture 6. *If $A = A_1, A_2, A_3, A_4$ admits an order 2 non-repeating Whitney tower $\mathcal{W} \subset X$, then the A_i can be pulled apart if and only if $\lambda_2(A) := \lambda_2(\mathcal{W})$ vanishes in $\Lambda_2(4)/INT_2(A)$.*

We want to describe $INT_2(A)$ in terms of the A_i and generators of $\pi_2 X$, using the lower order intersection trees λ_0 and λ_1, which are well-defined homotopy invariants. These $INT_2(A)$ relations will describe all indeterminacies due the choice of interiors of Whitney disks in $\mathcal{W}$, and if $\lambda_2(\mathcal{W}) \in INT_2(A)$, then the A_i can be pulled apart (Proposition 7.3 below). What remains to be proved in Conjecture 6 is that the vanishing of $\lambda_2(\mathcal{W})$ in $\Lambda_2(4)/INT_2(A)$ does not depend on the *boundaries* of the Whitney disks in $\mathcal{W}$.

Throughout this section we assume that the ambient 4-manifold X is simply connected.

7.1. **Bracket-labels and contractions.** It will be convenient to extend the labels on univalent vertices of trees in $\Lambda(m)$ to include bracketings from $\{1, 2, \ldots, m\}$. In non-repeating trees all indices (bracketed or not) must be distinct; e.g., a Y-tree with labels $(1, 2)$, 3, and 4 is non-repeating, but a Y-tree with labels $(1, 2)$, 2, and 3 is repeating. These bracket labels are temporary and will be “contracted” away: The *contraction* of a tree t along a univalent vertex v_I of t labelled by a bracket I is the tree $t(I) \cdot t$, where the inner product is taken by considering v_I as a root of t. Thus, a contraction just replaces a bracket labelled vertex with the subtree corresponding to the bracket.

We introduce these bracket-labels in order to use the following notational convention: Changing the interior of a Whitney disk W_I to W'_I (with $\partial W_I = \partial W'_I$) determines a 2-sphere S_I which is the union of W_I and W'_I (with the opposite orientation on W'_I). The resulting indeterminacies will be expressed as lower order intersection trees (containing the label I) which are contracted along the I-labeled vertices.

Before examining the order 2 INT relations it will be helpful to discuss the analogous lower order indeterminacies.

7.2. **First order intersection indeterminacies.** Recall (2.5) that the order zero non-repeating intersection tree $\lambda_0(A_1,\ldots,A_m)=\sum \epsilon_p\cdot i\!-\!\!-\!j\in\Lambda_0(m)$ is equivalent to the usual homological intersection form on $H_2(X)$, with the sum of the coefficients of each $i\!-\!\!-\!j$ corresponding to $[A_i]\cdot[A_j]$. There are no intersection indeterminacies in the order zero setting, and the collection $A_1,\ldots,A_m$ admits an order 1 non-repeating Whitney tower if and only if $\lambda_0(A_1,\ldots,A_m)$ vanishes in $\Lambda_0(m)$, (which is isomorphic to a direct sum of copies of $\mathbb{Z}$, one for each pair i,j of indices).

For A_1,A_2,A_3 with $\lambda_0(A_1,A_2,A_3)=0$, the order 1 non-repeating intersection tree $\lambda_1(A_1,A_2,A_3)$ is a sum of order one Y-trees in $\Lambda(3)\cong\mathbb{Z}$ modulo the $INT_1(A_1,A_2,A_3)$ intersection relations:

$$t(i,j)\cdot\lambda_0(S_{(i,j)},A_k)={}^{j}_{i}\!\vdash\lambda_0(S_{(i,j)},A_k)=0$$

where $S_{(i,j)}$ ranges over π_2X, and (i,j) ranges over the three choices of pairs from $\{1,2,3\}$. Geometrically, these relations correspond to tubing any Whitney disk $W_{(i,j)}$ into any 2-sphere $S_{(i,j)}$. Thinking of the $\lambda_0(S_{(i,j)},A_k)$ as integers, the quotient $\Lambda(3)/INT_1(A_1,A_2,A_3)$ is isomorphic to $\mathbb{Z}_d$, where d is the greatest common divisor of all the $\lambda_0(S_{(i,j)},A_k)$. This invariant $\lambda_1(A_1,A_2,A_3)\in\Lambda(3)/INT_1(A_1,A_2,A_3)$ is the *Matsumoto triple* ([26, 37]) which vanishes if and only if (after perhaps some finger moves) the A_1,A_2,A_3 admit an order 2 non-repeating Whitney tower.

Aside on computing the $INT_1(A_1,A_2,A_3)$ intersection relations: Let S^α be a basis for π_2X (modulo torsion), and define integers $a^\alpha_{ij}:=\lambda_0(S^\alpha_{(i,j)},A_k)$ for $S^\alpha_{(i,j)}=S^\alpha$ ranging over the basis. Then, identifying $\Lambda(3)\cong\mathbb{Z}$, all the $INT_1(A_1,A_2,A_3)$ intersection relations can be expressed as

$$\sum_\alpha(x^\alpha_{12}a^\alpha_{12}+x^\alpha_{31}a^\alpha_{31}+x^\alpha_{23}a^\alpha_{23})=0$$

where the x^α_{ij} vary over $\mathbb{Z}$. Thus, $INT_1(A_1,A_2,A_3)$ is the image of the linear map $\mathbb{Z}^r\oplus\mathbb{Z}^r\oplus\mathbb{Z}^r\longrightarrow\mathbb{Z}$ (r the rank of π_2X mod torsion) which can be written concisely using vector notation as:

$$(x_{12},x_{31},x_{23})\longmapsto x_{12}\cdot a_{12}+x_{31}\cdot a_{31}+x_{23}\cdot a_{23}.$$

For a collection A of *four* order zero surfaces A_1,A_2,A_3,A_4, with vanishing $\lambda_0(A)$, the order 1 non-repeating intersection tree $\Lambda_1(A)$ takes values in $\Lambda_1(4)/INT_1(A)$, and the $INT_1(A)$ relations are:

$$t(i,j)\cdot\lambda_0(S_{(i,j)},A_k,A_l)={}^{j}_{i}\!\vdash\lambda_0(S_{(i,j)},A_k,A_l)=0$$

with $S_{(i,j)}$ ranging over π_2X, and (i,j) ranging over the six choices of pairs from $\{1,2,3,4\}$. Here $\Lambda_1(4)\cong\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$, and each generator of π_2X gives six relations, so the target group $\Lambda(4)/INT_1(A)$ of the invariant $\lambda_1(A)$ is the quotient of Z^4 by the image of a linear map from $\mathbb{Z}^{6r}$, where r is the rank of $\pi_2 X$ modulo torsion. The invariant $\lambda_1(A)$ vanishes in $\Lambda(4)/INT_1(A)$ if and only if A admits an order 2 non-repeating Whitney tower (perhaps after some finger moves).

Note that each of the four copies of $\mathbb{Z}$ in $\Lambda_1(4)$ corresponds to a target of a Matsumoto triple, but the vanishing of the all the triples is not sufficient to get an order 2 non-repeating Whitney tower on the A because of "cross-terms" in the INT relations; the simplest example is the following: Consider a 4 component link in S^3 consisting of the Borromean rings together with a split fourth component. Attach a 4-dimensional 2-handle h along a circle which has linking number 1 with both the split component and one component of the Borromean rings. Now any immersed disks A_1, A_2, A_3, A_4 in $X = B^4 \cup h$ bounded by the link $L \subset \partial X$ have vanishing first order triples $\lambda(A_i, A_j, A_k)$, since the 2-sphere S determined by h is dual to (at least) one of A_i, A_j, A_k, but the first order quadruple $\lambda_1(A_1, A_2, A_3, A_4)$ is non-zero in $\Lambda(4)/INT_1(A_1, A_2, A_3, A_4) \cong \mathbb{Z}^3$. (Geometrically, the non-zero λ_1 of the disks bounded by the Borromean rings is killed by tubing S into a Whitney disk, but this creates an intersection between the Whitney disk and the disk bounded by the trivial component. Algebraically, this is seen in the collapsing of $\mathbb{Z}^4$ to $\mathbb{Z}^3$ by the INT relation due to S.)

Aside on computing the $INT_1(A_1, A_2, A_3, A_4)$ relations: Choose a basis S^α for $\pi_2 X$ modulo torsion, and define integers $a^\alpha_{ij,k} = \lambda_0(S^\alpha_{(i,j)}, A_k)$. Then each element of the subgroup $INT_1(A_1, A_2, A_3, A_4)$ can be written

$$\begin{aligned} {}^{j}_{i}\vdash \lambda_0(\textstyle\sum_\alpha x^\alpha_{ij} S^\alpha_{(i,j)}, A_k, A_l) &= (\textstyle\sum_\alpha x^\alpha_{ij} a^\alpha_{ij,k})\, {}^{j}_{i}\vdash k + (\textstyle\sum_\alpha x^\alpha_{ij} a^\alpha_{ij,l})\, {}^{j}_{i}\vdash l \\ &= (x_{ij} \cdot a_{ij,k})\, {}^{j}_{i}\vdash k + (x_{ij} \cdot a_{ij,l})\, {}^{j}_{i}\vdash l \end{aligned}$$

where the coefficients in the last expression are inner products of vectors in $\mathbb{Z}^r$. Using the basis

$$\{ {}^{1}_{2}\vdash 3,\ {}^{1}_{2}\vdash 4,\ {}^{1}_{3}\vdash 4,\ {}^{2}_{3}\vdash 4 \}$$

the subgroup $INT_1(A_1, A_2, A_3, A_4)$ is the image of the map $\mathbb{Z}^{6r} \longrightarrow \mathbb{Z}^4$:

$$\begin{pmatrix} x_{12} \\ x_{13} \\ x_{41} \\ x_{23} \\ x_{24} \\ x_{34} \end{pmatrix} \longmapsto \begin{pmatrix} a_{12,3} & -a_{13,2} & 0 & a_{23,1} & 0 & 0 \\ a_{12,4} & 0 & a_{41,2} & 0 & a_{24,1} & 0 \\ 0 & a_{13,4} & a_{41,3} & 0 & 0 & a_{34,1} \\ 0 & 0 & 0 & a_{23,4} & -a_{24,3} & a_{34,2} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{13} \\ x_{41} \\ x_{23} \\ x_{24} \\ x_{34} \end{pmatrix}$$

where the multiplication of entries is vector inner product.

7.3. **A relative first order invariant.** Consider order zero surfaces A_1, A_2, A_3, A_4 with $\lambda_1(A_1, A_2, A_3, A_4) = 0 \in \Lambda(4)/INT_1(A_1, A_2, A_3, A_4)$, so that A_1, A_2, A_3, A_4 admit an order 2 non-repeating Whitney tower. Let $A_{(1,2)}$ be any order zero surface satisfying $\Lambda_0(A_{(1,2)}, A_3, A_4) = 0$, so that $A_{(1,2)}, A_3, A_4$ admit an order 1 non-repeating Whitney tower. Note that $A_{(1,2)}$ may have non-trivial order 0 intersections with A_1 and A_2 which, as notation suggests, we will consider to be repeating intersections. We will define an invariant $\Lambda_1(A_{(1,2)}, A_3, A_4)$ (rel A_1, A_2) which is the complete obstruction to finding an order 2 Whitney tower on $A_{(1,2)}, A_3, A_4$ which also extends to an order 2 Whitney tower on A_1, A_2, A_3, A_4. To define this relative invariant we restrict to order 1 non-repeating Whitney towers on $A_{(1,2)}, A_3, A_4$ whose Whitney disks $W_{(3,4)}$ intersect A_1 and A_2 in canceling pairs. For instance, starting with any order 2 non-repeating Whitney tower on A_1, A_2, A_3, A_4, and adding Whitney disks pairing the order 0 intersections between $A_{(1,2)}$ and A_3, and between $A_{(1,2)}$ and A_4 yields such a Whitney tower. For any such Whitney tower $\mathcal{W}$, the invariant $\Lambda_1(A_{(1,2)}, A_3, A_4)$ (rel A_1, A_2) is defined in the usual way by counting the Y-trees associated to the order 1 non-repeating intersections in $\mathcal{W}$. Here the univalent labels are from $\{(1,2), 3, 4\}$ and the $INT_1(A_{(1,2)}, A_3, A_4)$ (rel A_1, A_2) relations are:

$$ {}^{(1,2)}_{\;\;3} \vdash \lambda_0(S_{((1,2),3)}, A_4) = 0 $$

$$ {}^{\;\;4}_{(1,2)} \vdash \lambda_0(S_{(4,(1,2))}, A_3) = 0 $$

$$ {}^{4}_{3} \vdash \lambda_0(S_{(3,4)}, A_{(1,2)}) = 0 $$

where $S_{((1,2),3)}$ and $S_{(4,(1,2))}$ range over all homotopy classes of 2-spheres, but $S_{(3,4)}$ ranges over only those which satisfy $\Lambda_0(A_1, A_2, S_{(3,4)}) = 0$ – since the Whitney disks in $\mathcal{W}$ which pair intersections between A_3 and A_4 are required to have canceling pairs of intersections with A_1 and A_2. The arguments of [30] also apply in this setting to show that this relative first order non-repeating intersection tree is a well-defined homotopy invariant, and is the complete obstruction to finding an order 2 Whitney tower on $A_{(1,2)}, A_3, A_4$ which also extends to an order 2 Whitney tower on A_1, A_2, A_3, A_4.

7.4. **Second order intersection indeterminacies.** Now, for a given order 2 non-repeating Whitney tower $\mathcal{W} \subset X$ on the four simply-connected order zero surfaces $A = \{A_1, A_2, A_3, A_4\}$, we will describe the $INT_2(A)$ relations which give all possible changes $\lambda_2(\mathcal{W}) - \lambda_2(\mathcal{W}') \in \Lambda_2(4)$ in the order 2 non-repeating intersection tree, where $\mathcal{W}'$ is gotten from $\mathcal{W}$ only by choosing different interiors of Whitney disks.

Note that $\Lambda_2(4)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, generated, for instance, by the elements

$$ z_1 := {}^{2}_{1}\!\vdash\!\dashv^{3}_{4}, \quad z_2 := {}^{3}_{1}\!\vdash\!\dashv^{2}_{4} $$

with

$$ {}^{4}_{1}\vdash\!\dashv^{2}_{3} = z_1 + z_2. $$

We first make some observations about order 2 Whitney disks, but we ultimately care about changes in the interiors of the order 1 Whitney disks, which completely change the order 2 Whitney disks.

7.4.1. ***Changing interiors of order 2 Whitney disks***. Let i, j, k, l be distinct indices from $\{1, 2, 3, 4\}$. Changing the interior of any order 2 Whitney disk $W_{((i,j),k)}$ determines a 2-sphere $S_{((i,j),k)}$, and the corresponding change in $\lambda_2(\mathcal{W})$ is equal to $t((i, j), k) \cdot t(l)$ times the order zero intersection number in $\mathbb{Z}$ between $S_{((i,j),k)}$ and A_l. This can be written as the contraction $t((i, j), k) \cdot \lambda_0(S_{((i,j),k)}, A)$, where the inner product is taken by considering the $((i, j), k)$-labelled univalent vertices in $\lambda_0(S_{((i,j),k)}, A)$ as roots.

Note that $\lambda_0(S_{((i,j),k)}, A)$ counts only intersections between $S_{((i,j),k)}$ and A_l, since $\lambda_0(A) = 0$, and because any intersections between $S_{((i,j),k)}$ and A_i, A_j, or A_k are repeating.

Letting $S_{((i,j),k)}$ vary over $\pi_2 X$, these intersection trees generate a subgroup $INT_{2,0}(A)$ of $\Lambda_2(4)$ (with the usual singleton labels only):

$$INT_{2,0}(A) := \langle t((i, j), k) \cdot \lambda_0(S_{((i,j),k)}, A)\rangle < \Lambda_2(4)$$

Note that $INT_{2,0}(A)$ is determined by a basis for $\pi_2 X$ modulo torsion, and that any element in $INT_{2,0}(A)$ can be realized (without any creating any other order 2 nonrepeating intersections) by creating clean Whitney disks $W_{((i,j),k)}$ which are then then tubed into 2-spheres (or just tubing into existing Whitney disks).

Definition 7.1. Denote by $\Lambda_{2,0}(A)$ the quotient of $\Lambda_2(4)$ by $INT_{2,0}(A)$.

Observe that $\Lambda_{2,0}(A) \cong \mathbb{Z}_d \oplus \mathbb{Z}_d$ where $d \in \mathbb{Z}$ is the greatest common divisor of $d_i = \lambda_0(S, A_i)$ over all S and i, (with $\Lambda_{2,0}(A) \cong \mathbb{Z} \oplus \mathbb{Z}$ if all $d_i = 0$). In particular, if at least one of the A_i has a dual 2-sphere, then all the A_i can be pulled apart:

Proposition 7.2. *If A_1, A_2, A_3, A_4 admit an order 2 non-repeating Whitney tower, and there exists $S \in \pi_2 X$ with $\lambda_0(S, A_i) = 1$ for any i, then the A_1, A_2, A_3, A_4 can be pulled apart. In particular, the A_1, A_2, A_3, A_4 can be pulled apart after connected summing X with a single $\mathbb{C}P^2$ or $S^2 \times S^2$.*

7.4.2. ***Changing interiors of order 1 Whitney disks***. Continue to let i, j, k, l denote distinct indices in $\{1, 2, 3, 4\}$. We want to describe all possible changes $\lambda_2(\mathcal{W}) - \lambda_2(\mathcal{W}') \in \Lambda_{2,0}(A)$ due to choices of interiors of order 1 Whitney disks.

Order 1 Whitney disks $W_{(i,j)}$ and $W'_{(i,j)}$ with common boundary determine a 2-sphere $S_{(i,j)}$ with $\lambda_0(S_{(i,j)}, A_k, A_l) = 0 \in \Lambda_0((i, j), k, l)$, since $\mathcal{W}$ and $\mathcal{W}'$ are order 2 (non-repeating) Whitney towers (Figure 11). The order 1 non-repeating intersection tree $\lambda_1(S_{(i,j)}, A_k, A_l)$ is defined in $\Lambda_1((i, j), k, l)$, modulo INT relations coming from order zero intersections which can be expressed as contractions:

$$ {}^{\,k}_{(i,j)} \vdash \lambda_0(S_{((i,j),k)}, A_l) = 0 \tag{1}$$

$$ {}^{(i,j)}_{\;\,l} \vdash \lambda_0(S_{(l,(i,j))}, A_k) = 0 \tag{2}$$

$$ {}^{l}_{k} \vdash \lambda_0(S_{(k,l)}, S_{(i,j)}) = 0. \tag{3}$$

We want to express the change $\lambda_2(\mathcal{W})-\lambda_2(\mathcal{W}') \in \Lambda_{2,0}(A)$ by contracting $\lambda_1(S_{(i,j)}, A_k, A_l)$ into $\Lambda_{2,0}(A)$ (by putting roots at the (i, j)-labelled vertices in $\lambda_1(S_{(i,j)}, A_k, A_l)$ and taking

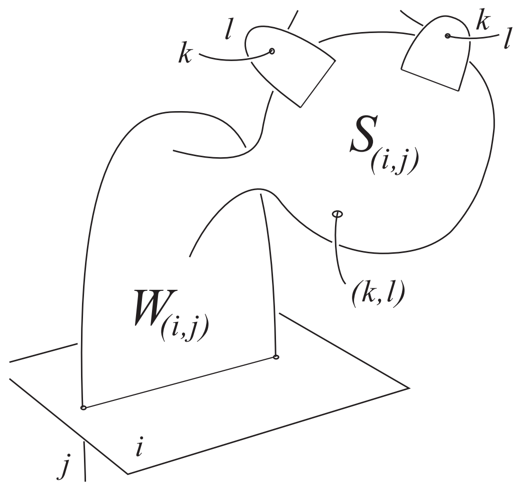

FIGURE 11. Changing the interior of $W_{(i,j)}$ to $W'_{(i,j)}$ corresponds to tubing $W_{(i,j)}$ into a 2-sphere $S_{(i,j)}$. Typical non-repeating intersections are shown; any repeating intersections are suppressed.

inner product with $t(i,j)$). Note that this contraction takes the first two relations (1) and (2) to zero in $\Lambda_{2,0}(A)$ (by the $INT_{2,0}(A)$ relations). The third relation (3) corresponds to indeterminacies in $\lambda_1(S_{(i,j)}, A_k, A_l)$ which are due to choices of Whitney disks pairing intersections between A_k and A_l; so if we ignore this relation (3) (since we are for the moment only allowing $W_{(i,j)}$ to change), then $\lambda_1(S_{(i,j)}, A_k, A_l)$ determines (via contraction) a well-defined element $t(i,j) \cdot \lambda_1^{\mathcal{W}}(S_{(i,j)}, A_k, A_l) \in \Lambda_{2,0}(A)$ which just depends on (the order 1 Whitney disks $W_{(k,l)}$ in) $\mathcal{W}$ and the homotopy class of $S_{(i,j)}$. Note that the contraction of $\lambda_1^{\mathcal{W}}(S_{(i,j)}, A_k, A_l)$ in $\Lambda_{2,0}(A)$ is the image of a lift (determined by $\mathcal{W}$) of the invariant $\lambda_1(S_{(i,j)}, A_k, A_l)$, which in the present setting can be taken to be the relative invariant (rel (A_i, A_j)) described above in subsection 7.3 (see next paragraph).

Now consider changing both $W_{(i,j)}$ to $W'_{(i,j)}$, and some $W_{(k,l)}$ to $W'_{(k,l)}$ as illustrated in Figure 12; recall that i,j,k,l are distinct. The resulting change $\Delta^{\mathcal{W}}(S_{(i,j)}, S_{(k,l)})$ of $\lambda(\mathcal{W})$ in $\Lambda_{2,0}(A)$ can be expressed

$$\Delta^{\mathcal{W}}(S_{(i,j)}, S_{(k,l)}) := {}^{j}_{i}\vdash \lambda_1^{\mathcal{W}}(S_{(i,j)}, A_k, A_l) + {}^{j}_{i}\vdash \lambda_0(S_{(i,j)}, S_{(k,l)}) \dashv^{k}_{l} + \lambda_1^{\mathcal{W}}(A_i, A_j, S_{(k,l)}) \dashv^{k}_{l} .$$

Here the 2-sphere $S_{(k,l)}$ determined by $W_{(k,l)}$ and $W'_{(k,l)}$ contributes intersections just as discussed for $S_{(i,j)}$, but now there is also a "cross-term" coming from (the contraction of) order zero intersections between $S_{(i,j)}$ and $S_{(k,l)}$; (as in the previous paragraph $\lambda_1^{\mathcal{W}}(S_{(i,j)}, A_k, A_l)$ and $\lambda_1^{\mathcal{W}}(A_i, A_k, S_{(k,l)})$ are lifts of the corresponding relative invariants). The important point here is that $\Delta^{\mathcal{W}}(S_{(i,j)}, S_{(k,l)})$ only depends on (the homotopy classes rel boundary of the Whitney disks and surfaces in) $\mathcal{W}$ and the homotopy classes of $S_{(i,j)}$ and $S_{(k,l)}$. Observe that, by the linearity of intersection trees, this entire discussion applies word for word to changing *all* the first order Whitney disks $W_{(i,j)}$ on A_i and A_j,

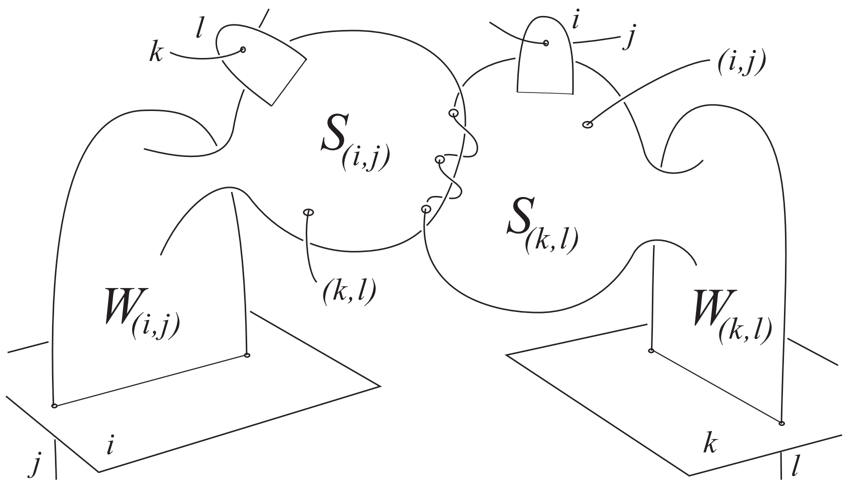

FIGURE 12. Changing the interiors of $W_{(i,j)}$ and $W_{(k,l)}$ corresponds to tubing into 2-spheres whose order zero intersections contribute order 2 indeterminacies. Sample non-repeating intersections are shown, and all repeating intersections are suppressed.

and *all* the first order Whitney disks $W_{(k,l)}$ on A_k and A_l; (with the 2-spheres $S_{(i,j)}$ and $S_{(k,l)}$ interpreted as sums (geometrically: disjoint unions) of the 2-spheres determined by each pair of Whitney disks).

Now letting $S_{(i,j)}$ and $S_{(k,l)}$ vary over all (homotopy classes of) 2-spheres such that $\lambda_0(S_{(i,j)}, A_k, A_l) = 0 \in \Lambda_0((i,j),k,l)$ and $\lambda_0(A_i, A_j, S_{(k,l)}) = 0 \in \Lambda_0((i,j),k,l)$, and letting (i,j), (k,l) vary over pair-choices $(1,2),(3,4)$, and $(1,3),(2,4)$ and $(1,4),(2,3)$, defines the sub*set*

$$INT_{2,1}^{\mathcal{W}}(A) := \cup\{-\Delta^{\mathcal{W}}(S_{(1,2)}, S_{(3,4)}) - \Delta^{\mathcal{W}}(S_{(1,3)}, S_{(2,4)}) - \Delta^{\mathcal{W}}(S_{(1,4)}, S_{(2,3)})\} \subset \Lambda_{2,0}(A)$$

which describes all possible changes $\lambda_2(\mathcal{W}) - \lambda_2(\mathcal{W}') \in \Lambda_2(4)$, where the first order Whitney disks in $\mathcal{W}'$ and $\mathcal{W}$ have the same boundaries.

Summing up the discussion so far, we have:

Proposition 7.3.

(i) *If $\lambda_2(\mathcal{W}) \in INT_{2,1}^{\mathcal{W}}(A)$, then the A_i can be pulled apart.*

(ii) *Assume that the order 1 Whitney disks in $\mathcal{W}$ and $\mathcal{W}'$ have the same boundaries. Then $\lambda_2(\mathcal{W}) \in INT_{2,1}^{\mathcal{W}}(A)$ if and only if $\lambda_2(\mathcal{W}') \in INT_{2,1}^{\mathcal{W}'}(A)$.*

(iii) *Assume that the order 1 Whitney disks in $\mathcal{W}$ and $\mathcal{W}'$ have the same boundaries. Then $INT_{2,1}^{\mathcal{W}}(A) = INT_{2,1}^{\mathcal{W}'}(A) \in \Lambda_{2,0}(A)$ if and only if $INT_{2,1}^{\mathcal{W}}(A)$ is closed under addition.*

Proof. Above discussion shows that we can create $+\Delta^{\mathcal{W}}(S_{(i,j)}, S_{(k,l)})$ by tubing...Etc... And, can check that if $\mathcal{W}'$ differs from $\mathcal{W}$ by $S'_{(i,j)}, S'_{(k,l)}$ and $\lambda_2(\mathcal{W}) = -\Delta^{\mathcal{W}}(S_{(i,j)}, S_{(k,l)})$, then

$$\lambda_2(\mathcal{W}') = -\Delta^{\mathcal{W}}(S_{(i,j)}, S_{(k,l)}) - \Delta^{\mathcal{W}}(S'_{(i,j)}, S'_{(k,l)}) = -\Delta^{\mathcal{W}'}(S_{(i,j)} - S'_{(i,j)}, S_{(k,l)} - S'_{(k,l)}).$$

□

Proposition 7.3 motivates the following definition:

Definition 7.4. For $A = \{A_1, A_2, A_3, A_4\}$ with vanishing $\lambda_1(A)$, define the order 2 non-repeating intersection tree

$$\lambda_2(A) := \lambda_2(\mathcal{W}) \in \Lambda_2(4)/INT_2(A)$$

where $\Lambda_2(4)/INT_2(A)$ denotes the quotient of $\Lambda_{2,0}(A)$ which identifies all elements of $INT^{\mathcal{W}}_{2,1}(A) \subset \Lambda_{2,0}$.

Note that $INT_{2,1}(A)$ always contains the zero element of $\Lambda_{2,0}$. If $INT^{\mathcal{W}}_{2,1}(A)$ is a subgroup of $\Lambda_{2,0}$, then $\Lambda_2(4)/INT_2(A)$ has a natural group structure (and $INT_{2,1}(A) \subset \Lambda_{2,0}(A)$ is independent of $\mathcal{W}$). In any case $\Lambda_2(4)/INT_2(A)$ has a well-defined zero element $INT_{2,1}(A) \subset \Lambda_{2,0}$, so the statement of Conjecture 6 makes sense.

In the case that all order zero intersections vanish on $\pi_2 X$, then ($INT_{2,0}(A)$ is trivial and) $INT_2(A) = INT_{2,1}(A)$ is a well-defined subgroup of $\Lambda_2(4)$ (independent of $\mathcal{W}$). If X is spin, then the same holds after tensoring with $\mathbb{Z}_2$ – in this case Conjecture 6 could still give an obstruction, but the vanishing would no longer guarantee that the A_i could be pulled apart.

(The understanding that $\lambda_2(A) := \lambda_2(\mathcal{W})$ and $INT^{\mathcal{W}}_{2,1}(A) \subset \Lambda_{2,0}$ are both computed using the same $\mathcal{W}$ can be thought of as being roughly analogous to choosing a basing...)

7.5. **Computing** $INT^{\mathcal{W}}_{2,1}(A)$**.** For 2-spheres S^α representing a minimal set of n generators for the $\mathbb{Z}$-module $\pi_2 X$, we have $\Lambda_{2,0}(A) \cong \mathbb{Z}_d \oplus \mathbb{Z}_d$ where $d \in \mathbb{Z}$ is the greatest common divisor of $d_i = \lambda_0(S^\alpha, A_i)$ over all α and i, with $\Lambda_{2,0}(A) \cong \mathbb{Z} \oplus \mathbb{Z}$ if all $d_i = 0$ (see 7.4.1 above).

Let $Q = q^{\alpha\beta} = \lambda_0(S^\alpha, S^\beta)$ denote the intersection matrix on $\pi_2 X$, including the zero intersection form on the torsion subgroup. For integers a^α_{ij} defined by

$$\lambda^{\mathcal{W}}_1(S^\alpha_{(i,j)}, A_k, A_l) = a^\alpha_{ij} \; {}^{j}_{i}\!\!\vdash\!\!\dashv^{k}_{l}$$

we have the formula

$$\begin{aligned}
\Delta^{\mathcal{W}}(\textstyle\sum_\alpha x^\alpha_{ij} S^\alpha_{ij}, \sum_\beta x^\beta_{kl} S^\beta_{kl}) &= \textstyle\sum_\alpha \Delta^{\mathcal{W}}(x^\alpha_{ij} S^\alpha_{ij}, x^\alpha_{kl} S^\alpha_{kl}) + \sum_{\alpha\neq\beta} \lambda_0(x^\alpha_{ij} S^\alpha_{ij}, x^\beta_{kl} S^\beta_{kl}) \\
&= \textstyle\sum_\alpha (x^\alpha_{ij} a^\alpha_{ij} + x^\alpha_{ij} x^\alpha_{kl} q^{\alpha\alpha} + x^\alpha_{kl} a^\alpha_{kl}) + \sum_{\alpha\neq\beta} x^\alpha_{ij} x^\beta_{kl} q^{\alpha\beta} \\
&= x_{ij} \cdot a_{ij} + x_{kl} \cdot a_{kl} + x_{ij} Q x^T_{kl}
\end{aligned}$$

where the x_{rs} and a_{rs} are vectors in $\mathbb{Z}^n$. Thus, computing $INT^{\mathcal{W}}_{2,1}(A)$ amounts to determining the image in $\Lambda_{2,0}(A)$ of the map:

$$\begin{pmatrix} x_{12}\\ x_{34}\\ x_{13}\\ x_{24}\\ x_{14}\\ x_{23}\end{pmatrix} \mapsto \begin{pmatrix} a_{12} & a_{34} & 0 & 0 & a_{14} & a_{23}\\ 0 & 0 & a_{13} & a_{24} & a_{14} & a_{23}\end{pmatrix}\begin{pmatrix} x_{12}\\ x_{34}\\ x_{13}\\ x_{24}\\ x_{14}\\ x_{23}\end{pmatrix} + \begin{pmatrix} x_{12}Qx_{34}^T + x_{14}Qx_{23}^T\\ x_{13}Qx_{24}^T + x_{14}Qx_{23}^T\end{pmatrix}$$

where the multiplication of entries is vector inner product.

Note that, since Q vanishes on torsion, the quotient of $\Lambda_{2,0}(A)$ by all the elements $\Delta^{\mathcal{W}}(S^\alpha_{ij}, S^\beta_{kl})$ where either of S^α_{ij} or S^β_{kl} are *torsion* classes is again isomorphic to $\mathbb{Z}_d \oplus \mathbb{Z}_d$ (or $\mathbb{Z}\oplus\mathbb{Z}$), where the greatest common divisor $d \in \mathbb{Z}$ now also depends on the a^γ_{rs}.

Thus, the essential difficulty in computing the image of the above map reduces to consideration of the case where the S^α are a basis for $\pi_2 X$ modulo torsion, and solving a collection of quadratic equations over $\mathbb{Z}$ (modulo d).

7.5.1. *Example.* In this example we consider the computation of $\Lambda_2(4)/INT_2(A)$ in the case $\pi_2 X = \langle S\rangle \cong \mathbb{Z}$ with self-intersection number $\lambda_0(S, S') = q \neq 0 \in \mathbb{Z}$. As before, A denotes the collection A_1, A_2, A_3, A_4 of order zero surfaces; and $\pi_1 X = 1$, with $\lambda_1(A) = 0 \in \Lambda_1(4)/INT_1(A)$, so that A admits an order 2 non-repeating Whitney tower $\mathcal{W}$. A basic case to keep in mind is where the A_i are properly immersed disks (rel ∂) FIGURE?.

First we note:

Lemma 7.5. *If $d_i := \lambda_0(S, A_i) \neq 0 \in \mathbb{Z}$ for at least one i, then $\Lambda_2(4)/INT_2(A)$ does admit a natural group structure.*

Proof. Using the above notation, the $INT_{2,0}$ relations give $\Lambda_{2,0}(A) = \mathbb{Z}_d \oplus \mathbb{Z}_d$ where d is the greatest common divisor of the d_i.

If $d_{i_0} \neq 0$ for exactly one i_0, then the only $INT_{2,1}$ relations are

$$\substack{j\\ i_0} \vdash \lambda_1(S_{(i_0,j)}, A_k, A_l) = 0$$

where, in this case, $\lambda_1(S_{(i_0,j)}, A_k, A_l)$ is a well-defined invariant – independent of $\mathcal{W}$ – because $d_{i_0} = \lambda_0(S, A_{i_0}) \neq 0$ means that S can not be tubed into any $W_{(k,l)}$. (Since all $W_{(k,l)}$ Whitney disks are required to have vanishing order intersections with A_{i_0} and A_j, this invariant is $\lambda_1(S_{(i_0,j)}, A_k, A_l)$ rel (A_{i_0}, A_j); i.e. it is the complete obstruction to finding an order 1 non-repeating Whitney tower on $S_{(i_0,j)}, A_k, A_l$ which is also an order 2 tower on A (7.3).

Similarly, if only d_{i_0} and d_{j_0} are non-zero then there is just a single $INT_{2,1}$ relation:

$$\substack{j_0\\ i_0} \vdash \lambda_1(S_{(i_0,j_0)}, A_k, A_l) = 0$$

where again $\lambda_1(S_{(i_0,j_0)}, A_k, A_l)$ is independent of $\mathcal{W}$.

Finally, if three or four of the d_i are non-zero, then there are no $INT_{2,1}$ relations, and so $\Lambda_2(4)/INT_2(A) = \Lambda_{2,0}(A)$.

□

For the rest of this subsection we assume that all d_i are zero, so that $INT_{2,0}(A)$ is trivial, and $\Lambda_{2,0}(A) = \Lambda_2(4)$. The $INT_{2,1}(A)$ indeterminacy subset in $\Lambda_2(4) \cong \mathbb{Z} \oplus \mathbb{Z}$ is equal tothe image of the map $\mathbb{Z}^6 \to \mathbb{Z}^2$ given by:

$$\begin{pmatrix} x_{12} \\ x_{34} \\ x_{13} \\ x_{24} \\ x_{14} \\ x_{23} \end{pmatrix} \mapsto \begin{pmatrix} a_{12} & a_{34} & 0 & 0 & a_{14} & a_{23} \\ 0 & 0 & a_{13} & a_{24} & a_{14} & a_{23} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{34} \\ x_{13} \\ x_{24} \\ x_{14} \\ x_{23} \end{pmatrix} + q \begin{pmatrix} x_{12}x_{34}^T + x_{14}x_{23}^T \\ x_{13}x_{24}^T + x_{14}x_{23}^T \end{pmatrix}$$

Try to solve in general using Mathematica...Are these equations solvable? Can image not be closed under addition? What about general case where equations are coupled by intersection matrix?...

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