Ideals

$R = (0, +, 1, \times)$

$I \subseteq R$

↳ closed under $+$

↳ $a \in I,\ b \in R \Rightarrow ab \in I$ (right ideal)

↳ $a \in R,\ b \in I \Rightarrow ab \in I$ (left ideal)

If $I \leq R$ is a 2-sided ideal

then $\exists\, \varphi : R \to S$ s.t. $I = \ker \varphi$

$R \to R/I$

kernels of ring homomorphisms $\Longleftrightarrow$ two-sided ideal

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What are the two-sided ideals in the ring of matrices $M_n(F)$?

$0 \neq I \leq M_n(F) \quad 0 \neq M \in I$

$$M = \begin{bmatrix} a_{11} & \cdots & \\ \vdots & a_{ij} \neq 0 & \\ \vdots & & a_{nn} \end{bmatrix}$$

$$\begin{bmatrix} 1_{(i,i)} \end{bmatrix} M \begin{bmatrix} 1_{(j,j)} \end{bmatrix} = \begin{bmatrix} 0 & a_{ij} \neq 0 & 0 \\ 0 & & 0 \end{bmatrix}$$

rescale $\rightsquigarrow \begin{bmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{bmatrix}$

$$\begin{bmatrix} & 1_{(1,i)} & \end{bmatrix}\begin{bmatrix} & 1_{(i,j)} & \end{bmatrix}\begin{bmatrix} & & 1_{(j,1)} \end{bmatrix}$$
$$= \begin{bmatrix} 1_{(1,1)} & \\ & \end{bmatrix}$$

I contains $\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & \ddots \\ & & & & 0 \end{bmatrix}$

So I contains every matrix

If $\varphi : M_n(F) \to R$, $\ker(I) = 0$ or $\ker(I) = M_n(F)$

φ injective

$\ker(I) = M_n(F)$

$\varphi(M) = 0$

$\varphi(1) = 1$

(either no such φ's, or only $M_n(F) \xrightarrow{0} R$)

unital ring

$m, n \geq 1$, $\underbrace{(x^m - 1, x^n - 1)}_{\text{Ideal generated}} \leq \underbrace{\mathbb{Z}[x]}_{\text{integer polynomials}}$

$= \{a(x^m - 1) + b(x^n - 1) : a, b \in \mathbb{Z}[x]\}$

If $n \geq m$,

x^{n-m}

If $n \geq m$,

$$x^n - 1 = x^{n-m}(x^m - 1) + x^{n-m} - 1$$

$$(x^m - 1, x^n - 1) = (x^m - 1, x^{n-m} - 1)$$

$\rightsquigarrow$ eventually: one exponent will reach 0

$(x^k - 1, x^0 - 1) = (x^k - 1)$ is principal

$(k = \gcd(m,n))$

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field $F[x]$ euclidean domain

UFD $\mathbb{Z}[x]$ UFD $\mathbb{Z}[x]$ not PID

Field $\subsetneq$ Euclidean Domains $\subsetneq$ Principal ideal domain

a, b $N(r) < N(b)$ Every ideal

$a = bq + r$ principal

$\subsetneq$ Unique Factorization Domain

$x \neq 0$ $x = p_1 \cdots p_r$ irred

unique up to permutation and multiplication by units

$\subsetneq$ Integral domain

$$R = \{a + 3bi : a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$$

integral domain: if $xy = 0$ then $x = 0$ or $y = 0$

(true since $R \subseteq \mathbb{C}$)

not UFD: $(a+3bi)(a-3bi) = a^2 + 9b^2$

$a = 1, b = 1,$

$(1+3i)(1-3i) = 10$

$= 2 \cdot 5$

Check: $1+3i$ irred

$= 2 \cdot \cdot$

Check: $1+3i$ irred

$2, 5$ irred

Or: $(3i)(-3i) = 3 \cdot 3$

Check: $3i$ irred, 3 irred ✓

Check: $3, 3i$ don't differ by a unit ($i \notin R$)

If $3 = ab$

then $|3|^2 = |a|^2 |b|^2$

$9 = \text{integer} \cdot \text{integer}$

$1 \cdot 9$

$3 \quad 3$

$a^2 + 9b^2 = 1$

$\Rightarrow a = \pm 1$

$b = 0$

$|a + 3bi|^2 \neq 3$

$a^2 + 9b^2 \neq 3$

R Field $\Longleftrightarrow$ (0) maximal

R Integral domain $\Longleftrightarrow$ (0) <u>prime</u>

$xy \in (0) \Rightarrow x \in (0)$ or $y \in (0)$

R/I field $\Longleftrightarrow$ I maximal

R/I integral domain $\Longleftrightarrow$ I prime

$R(X, F) = \{ f : \underset{\text{finite set}}{X} \longrightarrow \underset{\text{field}}{F} \}$ pointwise $+, \times$

maximal ideals?

$\varphi : R(X, F) \twoheadrightarrow F'$ (since then $\frac{R(X,F)}{\ker} \cong F'$ field $\Rightarrow \ker$ maximal)

$1_a(x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases}$

$a \in X$

$$1_a \cdot 1_b = 0$$

$$\varphi(1_a \cdot 1_b) = 0$$

$$\varphi(1_a) \cdot \varphi(1_b) = 0$$

so $\varphi(1_a) = 0$ or $\varphi(1_b) = 0$.

~~at most~~ exactly one 1_a s.t. $\varphi(1_a) \neq 0$.

$f \in \ker \varphi \iff \varphi(f) = 0$

$\iff \varphi(f)\varphi(1_a) = 0$ $\to f(a) \cdot 1_a$

$\iff \varphi(f \cdot 1_a) = 0$

$\iff f(a) = 0$

$\iff f \in \underline{\ker(ev_a : R(X,F) \to F)}$

$\varphi(f(a) \cdot 1_a) = 0$

if $f(a) \neq 0$: $\varphi(f(a)^{-1} \cdot f(a) \cdot 1_a) = 0$

$\varphi(1_a) = 0$

contradiction