# Real Analysis

In $\mathbb{R}^n$, compactness $\Longleftrightarrow$ Every open cover has finite subcover

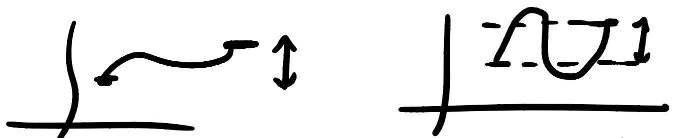
(cont) $f$(compact) = compact $\Longleftrightarrow$ If $x_n \subseteq K$, then there is a convergent subseq

$\Longleftrightarrow$ $K$ is closed and bounded

$S$ Connected means $S$ cannot be written as
$\underset{\text{nontrivial}}{S = X \cup Y}$ with $\overline{X} \cap Y = \emptyset$, $X \cap \overline{Y} = \emptyset$

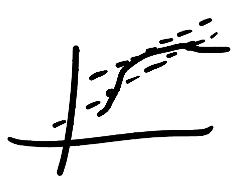
image of connected space under continuous map remains connected

$\underset{\text{connected}}{f: X} \longrightarrow \mathbb{R}$, if $a, b \in \text{image}(f)$
then so is every $a < c < b$



Intermediate Value Theorem

Mean Value Theorem $f: [a,b] \to \mathbb{R}$
cont on $[a,b]$, diff on $(a,b)$
$\exists\, c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$



Rolle's Theorem: If $f(a) = f(b)$, then
$\exists\, c \in (a,b)$ s.t. $f'(c) = 0$

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a) No surj $\underset{\text{compact}}{[0,1]} \twoheadrightarrow \underset{\text{not compact}}{(0,1)}$

b) Surj $(0,1) \to [0,1]$

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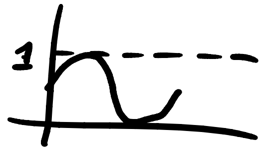
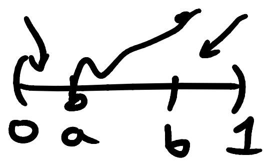



$\frac{1}{2}+\frac{1}{2}\sin(2\pi x)$

c) No bij $(0,1) \to [0,1]$

look at $f(a)=0$, $f(b)=1$




then by IVT

$[a,b] \twoheadrightarrow [0,1]$

so where does $\frac{a}{2}$ go?

$[0,2\pi) \to \{|z|=1\}$

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$f$ cont. real valued on $[0,1]$

Show $\int_0^1 f(x)x^2\,dx = \frac{1}{3}f(\xi)$

some $\xi \in [0,1]$

Look at $\frac{1}{3}f(\xi) - \int_0^1 f(x)x^2\,dx$

$\int_0^1 f(x)x^2\,dx \le f(\xi_{max})\frac{1}{3}$

$\le$

$f(\xi_{min})\frac{1}{3}$

$= \int_0^1 f(\xi)x^2\,dx - \int_0^1 f(x)x^2\,dx$

$= \int_0^1 (f(\xi)-f(x))\,x^2\,dx$

$\xi_{max}$ makes this $\ge 0$

$\xi_{min}$ makes this $\le 0$

some $\xi$ between them makes $=0$

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Bounded monotone sequence must converge

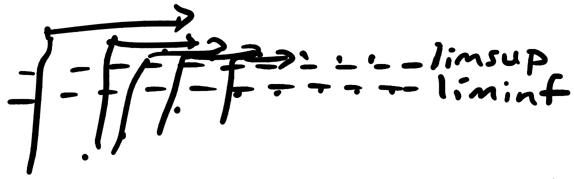
$\{x_n\}_{n=1}^{\infty}$ $\quad x_1 \le x_2 \le x_3 \le \cdots \le B$

$\Rightarrow$ must converge

$n \mapsto \sup\limits_{k\ge n} x_n$ $\quad$ non-increasing $\longrightarrow \limsup x_n$

$n \mapsto \inf\limits_{k\ge n} x_n$ $\quad$ non-decreasing $\longrightarrow \liminf x_n$

$n \mapsto \inf_{k \geq n} x_n$ non-decreasing $\longrightarrow \liminf x_n$




$\{x_n\}$ converges $\Longleftrightarrow$ liminf = limsup

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$\{x_n\}$ $\quad 2x_{n+1} - x_n \longrightarrow x$, show $x_n \to x$

$$x_{n+1} = \frac{1}{2}(x_n + (2x_{n+1} - x_n))$$

$$\limsup x_n = \limsup x_{n+1} \leq \frac{1}{2}(\limsup x_n + x)$$

$$\liminf x_n = \liminf x_{n+1} \geq \frac{1}{2}(\liminf x_n + x)$$

$\limsup x_n \leq x$

$\liminf x_n \geq x$

$\limsup x_n \leq x \leq \liminf x_n$

also: $\liminf x_n \leq \limsup x_n$

so $\liminf x_n = \limsup x_n = x$

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$a, x_0 > 0$, $\quad x_n = \frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right)$

if $x_n \longrightarrow L$, then $L = \frac{1}{2}\left(L + \frac{a}{L}\right)$

$\limsup \leq \frac{1}{2}\left(\limsup + \frac{a}{\liminf}\right)$

$\liminf \geq \frac{1}{2}\left(\liminf + \frac{a}{\limsup}\right)$

$\limsup \leq \frac{a}{\liminf}$

$\liminf \geq \frac{a}{\limsup}$

$\liminf \cdot \limsup = a$

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right)$$

$$x_n = \frac{1}{2}\left(x_{n-1} - 2\sqrt{a} + \frac{a}{x_{n-1}}\right) + \sqrt{a}$$

$$= \frac{1}{2}\left(\sqrt{x_{n-1}} - \sqrt{\frac{a}{x_{n-1}}}\right)^2 + \sqrt{a}$$

$$\geq \sqrt{a}$$

$2L = L + \frac{a}{L}$

$L = \frac{a}{L}$

$a = L^2$

$L = \sqrt{a}$

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Or! $\frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right) \geq \sqrt{a}$

$$\left(x_{n-1} + \frac{a}{x_{n-1}}\right)^2 \geq 2a$$

$$x_{n-1}^2 + \frac{a^2}{x_{n-1}^2} \geq 0$$

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Show $x_n < x_{n-1}$

Show $\frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right) \leq x_{n-1}$

$$\frac{a}{x_{n-1}} \leq x_{n-1}$$

$$a \leq x_{n-1}^2$$

$$x_{n-1}^2 + \frac{a^2}{x_{n-1}^2} \geq 0$$

$$a \leq x_{n-1}^2$$

$$x_{n-1}^2 \geq \sqrt{a}$$

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$$L^2([a,b]) = \{ f : [a,b] \to \mathbb{C} \mid \int_a^b |f|^2 < \infty \}$$

Inner product: $\langle f, g \rangle = \int_a^b f \bar{g}$, $\|f\| = \langle f, f \rangle^{\frac{1}{2}} = \left( \int_a^b |f|^2 \right)^{\frac{1}{2}}$

$$|\langle f, g \rangle| \leq \|f\| \cdot \|g\|$$

$$\left| \int_a^b f \bar{g} \right| \leq \left( \int_a^b |f|^2 \right)^{\frac{1}{2}} \left( \int_a^b |g|^2 \right)^{\frac{1}{2}} \quad \text{equality} \Leftrightarrow f = \lambda g$$

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$0 \leq a \leq 1$, nonneg $f$ on $[0,1]$

$$\int_0^1 f(x)\,dx = 1, \quad \int_0^1 x f(x)\,dx = a, \quad \int_0^1 x^2 f(x)\,dx = a^2$$

$$\left| \int_a^b \sqrt{f}\, x\sqrt{f} \right| \leq \left( \int_a^b f \right)^{\frac{1}{2}} \left( \int_a^b x^2 f \right)^{\frac{1}{2}}$$

$a \leq 1 \cdot a$ equality

$\sqrt{f} = \lambda x \sqrt{f} \Rightarrow f = 0$ (contradicts $\int_0^1 f = 1$)

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Taylor's Thm: $f$ $k+1$ times diff on $(a,x)$

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \cdots + \frac{1}{k!} f^{(k)}(a)(x-a)^k + R_k(x)$$

$$R_k(x) = \frac{1}{(k+1)!} f^{(k+1)}(\xi)(x-a)^{k+1} \text{ some } \xi \in (a,x).$$

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$f : \mathbb{R}^n \to \mathbb{R}$ $\quad \nabla f = (\partial_1 f, \ldots, \partial_n f)$

$\nabla f = \vec{0}$ means critical point

Hessian matrix $(\partial_i \partial_j f)_{ij}$ symmetric

$2 \times 2$

pos def $\Rightarrow f$ local min, neg def $\Rightarrow f$ local max $\Big\}$ det $> 0$, $tr = f_{xx} + f_{yy} = \lambda_1 + \lambda_2$

$\Rightarrow f$ saddle pt $\}$ det $< 0$

neg def $\Rightarrow f$ local max, $tr = f_{xx} + f_{yy} = \lambda_1 + \lambda_2$

both pos, neg $\Rightarrow f$ saddle pt $\}$ det$<0$

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$f: \mathbb{R}^m \to \mathbb{R}^n$

Jacobian $(\partial_j f_i)_{ij}$

Inverse Function Thm: $J$ invertible (some $\vec{x}$)
then $f: U \to V$ bij with
differentiable inverse