Similarity: Two linear transformations or matrices $/F$ are similar if there exists a change of basis matrix P such that $M = P^{-1} N P$

Diagonalizable:
$$\begin{matrix} Pe_1 \\ \vdots \\ Pe_n \end{matrix} \xleftarrow{P} \begin{matrix} e_1 \\ \vdots \\ e_n \end{matrix}$$
Similar to a diagonal matrix (equivalently, $\exists$ basis of eigenvectors)

R a PID, module M (additive group action of R
$r(m+n) = rm + rn$
$(r+s)m = rm + sm$
$(rs)m = r(sm)$)

If M finitely generated (some $m_1, \dots, m_k$ such that $Rm_1 + \cdots + Rm_k = M$)

(Rational Canonical Form) (Invariant Factor Decomposition)

$$M \cong R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_k)$$

unique up to multiplication by a unit $\to$ $a_1 \mid a_2 \mid \cdots \mid a_k$ (last few could be zero)

unique $\to$ $(a_1) \supseteq (a_2) \supseteq \cdots \supseteq (a_k)$

$$\text{PID} \Rightarrow \text{UFD} \quad a_i = u \cdot \underbrace{p_1^{b_1} \cdots p_j^{b_j}}_{\text{prime/irreducible}}$$

CRT: $R/(a_i) \cong R/(p_1^{b_1}) \oplus \cdots \oplus R/(p_j^{b_j})$

Primary decomposition: $M \cong R/(p_?^?) \oplus \cdots \oplus R^r$

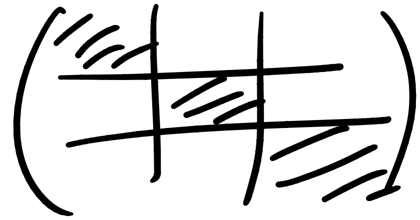
$T: V \to V$ fin dim view V/F as $F[x]$-module

$(a_n x^n + \cdots + a_1 x + a_0)v = a_n T^n v + \cdots + a_1 T v + a_0 v$

$/F$

$(a_n x^n + \cdots + a_1 x + a_0)v = a_n T^n v + \cdots + a_1 T v + a_0 v$

$V \cong F[x]/(f_1) \oplus \cdots \oplus F[x]/(f_k)$



f_i's polynomials$/F$

$f_1 \mid f_2 \mid \cdots \mid f_k$

How does T act on $\underbrace{F[x]/(f)}_{\text{basis } \{1, x, x^2, \dots, x^{n-1}\}}$?

$f = a_0 + a_1 x + \cdots + x^n$

$T \cdot x^{n-1} = x^n \equiv -a_{n-1}x^{n-1} - a_{n-2}x^{n-2} - \cdots - a_1 x - a_0$

$$\begin{pmatrix} 0 & 0 & & & -a_0 \\ 1 & 0 & & 0 & -a_1 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 \cdots & 1 & -a_{n-1} \end{pmatrix}$$ Companion matrix of f

RCF: T similar to
$$\begin{pmatrix} f_1 & & \\ & f_2 & \\ & & f_3 \end{pmatrix}$$
matrix with entries in F

$\underbrace{f_1 \mid f_2 \mid f_3 \mid \cdots \mid}_{\substack{\text{product} \\ = \text{characteristic} \\ \text{polynomial}}} f_n$

minimal polynomial of T

Jordon form: assume $F = \bar{F}$.

primary decomposition has $\dfrac{F[x]}{(x-\lambda)^k}$ $\quad (x-\lambda)^{k-1}, \dots, (x-\lambda), 1$

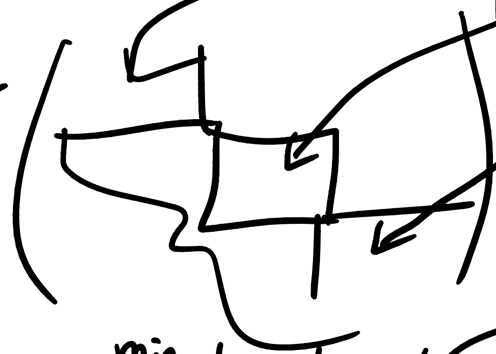
$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \end{pmatrix} \quad k \times k$$

decomposition has $(x-\lambda)^{\cdot\cdot}$

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \quad k \times k$$

minimal polynomial
$=$ characteristic polynomial
$= (x-\lambda)^k$

T similar to



minpoly = lcm
charpoly = product

Real matrices

Similar over $\mathbb{C}$ $\Longrightarrow$ Similar over $\mathbb{R}$

Have real RCF, but real RCF will be valid complex RCF

similar over $\mathbb{C}$ $\Longrightarrow$ same complex RCF
$\Longrightarrow$ same real RCF

Diagonalizable $\Longleftrightarrow$ minpoly factors into linear factors with no repeated roots

Triangularizable $\Longleftrightarrow$ minimal polynomal (or charpoly) factors into linear factors.