Linear Algebra 1

Outline

Fundamentals

- Linear independence, bases, dimension
- Rank-Nullity
- Eigenvalues/diagonalizability
- Inner products and spectral theorem

Determinants

- Cofactor expansion
- Multilinear/alternating
- Triangular case

Characteristic and Minimal Polynomial

- Cayley-Hamilton
- Trace and determinant from characteristic polynomial

Problems

7.1.1 Let p, q, r, s be polynomials of degree at most 3. Which, if any, of the following two conditions is sufficient for the conclusion that the polynomials are linearly dependent?

- At 1, each of the polynomials has value 0.
- At 0 each of the polynomials has value 1.

7.2.7. Let T be a real symmetric $n \times n$ matrix with upper and lower diagonal having entries $b_1, \dots, b_{n-1}$, and diagonal entries $a_1, \dots, a_n$, all other entries zero (it is a *tridiagonal matrix*). Assume $b_j \neq 0$ for all j. Show that the rank of T is at least $n - 1$ and that T has n distinct eigenvalues.

7.2.11. Let $\mathbb{R}[x_1, \dots, x_n]$ be the polynomial ring over the real field $\mathbb{R}$ in the n variables $x_1, \dots, x_n$. Let the matrix A be the $n \times n$ matrix whose ith row is $(1, x_i, x_i^2, \dots, x_i^{n-1})$, $i = 1, \dots, n$. Show that

$$\det(A) = \prod_{i>j} (x_i - x_j).$$

7.2.12. Consider an $(n + 1) \times (n + 1)$ matrix such that the ith row is $(1, a_i, a_i^2, \dots, a_i^n)$, $i = 0, \dots, n$ where the a_i are complex numbers. Prove that this matrix is invertible if the a_i are all different. (Do not use 7.2.11). In this case, prove that for any n complex numbers $b_0, \dots, b_n$, there exists a unique complex polynomial f of degree n such that $f(a_i) = b_i$ for $i = 0, \dots, n$.

7.5.13. Let F be a field, n and m positive integers, and A and $n \times n$ matrix with entries in

F such that $A^m = 0$. Prove that $A^n = 0$.

7.6.5. Compute A^{10} for the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$