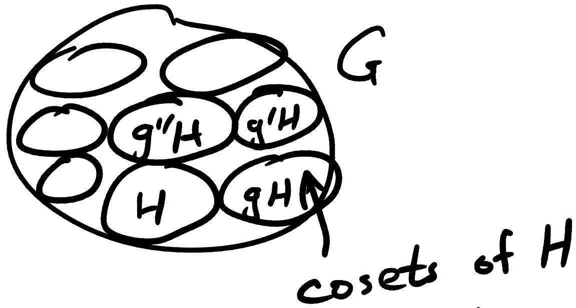
Group G

Subgroup H

index $[G:H] = \dfrac{|G|}{|H|}$



$[G:H] =$ # of cosets of H

Homomorphisms $\varphi: G \longrightarrow H$

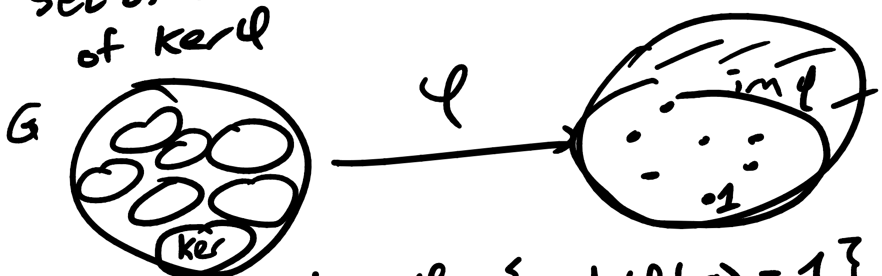
$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$

$\Longrightarrow \varphi(1) = 1$

$\Longrightarrow \varphi(g^{-1}) = \varphi(g)^{-1}$

First isomorphism theorem

$$\underbrace{G/_{\ker\varphi}}_{\text{set of cosets of } \ker\varphi} \cong \operatorname{im}\varphi$$



$\ker\varphi = \{g \mid \varphi(g) = 1\}$

Cyclic C_n, $\mathbb{Z}/n\mathbb{Z}$, n elements

$\{0, 1, \dots, n-1\}$ mod n

Automorphisms - generated by 1

$1 \longmapsto k$ coprime to n - generated by each of the $\varphi(n)$ elements coprime to n

$\mathrm{Aut}(C_n) \cong (\mathbb{Z}/n\mathbb{Z})^\times$

Dihedral groups: Symmetries of an n-gon

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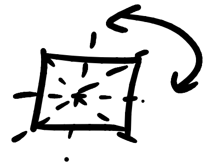
D_n vs D_{2n}



$D_n \cong C_n \rtimes C_2$ n rotations $\rightsquigarrow$ subgroup of index 2

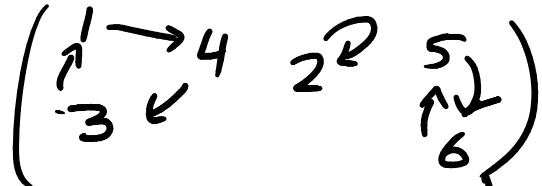
n reflection $\rightsquigarrow$ each has order 2

$C_2 \to \mathrm{Aut}(C_n)$ $g^2 = 1$

Symmetric groups: Permutations of a set of size n

$|S_n| = n!$ $\{1, 2, \ldots, n\}$

$\sigma \in S_n$



cycle type

$\downarrow$

$1+2+3 \leftarrow (143)(2)(56)$

$[421365]$ has order 6

$\sigma, \tau \in S_n$

$\tau = g^{-1}\sigma g$ iff same cycle type

sign: $S_n \to \{\pm 1\}$ – counts the parity of the number of transpositions (ij) required to form σ.

$1 \rightleftarrows 2 \to 3 \to 4$

$\det\begin{bmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \end{bmatrix}$ – parity of $\{(i,j) \mid i<j, \sigma(i) > \sigma(j)\}$

$\ker \mathrm{sign} = A_n$ alternating group, $|A_n| = \frac{n!}{2}$

For $n \geq 5$, A_n has no normal subgroup (kernel of some homomorphisms) (or $gHg^{-1} = H, \forall g$)

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invertible $n \times n$

homomorphisms,

Matrix Groups $GL(n,p) \to$ invertible $n\times n$ matrices over $\mathbb{F}_p$

normal

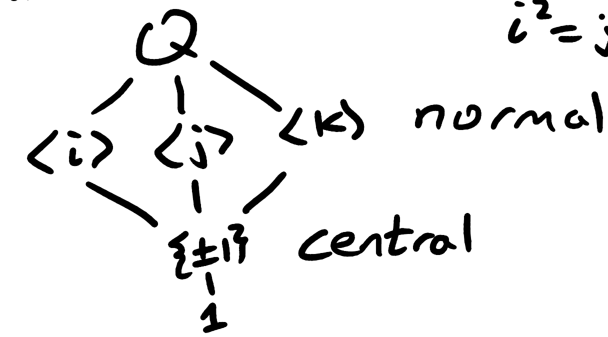
$SL(n,p) \to$ determinant 1 $n\times n$ matrices over $\mathbb{F}_p$

$$SL(n,p) = \ker(\det : GL(n,p) \twoheadrightarrow \mathbb{F}_p^\times)$$

$$|GL(n,p)| = (p^n-1)(p^n-p)(p^n-p^2)\cdots(p^n-p^{n-1})$$

Quaternion $Q = \{\pm 1, \pm i, \pm j, \pm k\}$

$i^2 = j^2 = k^2 = -1$, $ij = k$, $ji = -k$



Group actions $G \curvearrowright X$

$G \times X \to X$

$\mathrm{Stab}(x) = \{g \mid g\cdot x = x\}$ subgroup of G

$1\cdot x = x$

$\mathrm{Orbit}(x) = \{g\cdot x \mid g\in G\}$ $\quad (gh)\cdot x = g\cdot(h\cdot x)$

Orbit-stabilizer Theorem

bijection

$$G/\mathrm{Stab}(x) \;\hat{=}\; \mathrm{Orbit}(x)$$

$$\frac{|G|}{|\mathrm{Stab}(x)|} = |\mathrm{Orb}(x)| \qquad \sum^{*} |\mathrm{Orb}(x)| = |X|$$

Conjugation action

$G \curvearrowright G$ $\quad g\bullet h = ghg^{-1}$

$$\mathrm{Stab}(h) = \{g \mid g\bullet h = h\}$$

$$\mathrm{Stab}(h) = \{ g \mid g \cdot h = h \}$$
$$= \{ g \mid ghg^{-1} = h \}$$
$$= \{ g \mid gh = hg \} = C_G(h)$$

centralizer

$\mathrm{Orb}(g)$ = conjugacy class of g

$$|\text{Conj class}| = \frac{|G|}{|C_G(g)|}, \quad \sum^{*} |\text{Conj class}| = |G|$$

$$\sum^{*} \frac{|G|}{|C_G(g)|} = |G|, \qquad |Z(G)| + \sum^{**} \frac{|G|}{|C_G(g)|} = |G|$$

(class equation)

center

$\{ g \mid \forall h, gh = hg \}$

Sylow's Theorems: $|G| = p^k m \quad p \nmid m$

Def: A Sylow p-subgroup is a subgroup of order p^k.

p^k
|
⋮
|
p^j
|
p^{j-1}
|
⋮
|
p^0

1) They exist ($\exists$ Sylow p-subgroup)
2) They're conjugate (If P, Q Sylow p-subgroups then $Q = g^{-1}Pg$)
3) n_p = # Sylow subgroups

$$n_p \equiv 1 \pmod p, \quad n_p = \frac{|G|}{|N_G(P)|} \Big| \frac{|G|}{|P|} = m$$

$$N_G(P) = \{ g \mid g^{-1}Pg = P \}$$

Group of order 30 has a (normal) cyclic subgroup of order 15.

$\boxed{\text{Sec 6.2 of Dummit \& Foote}}$

$n_3 | 10 \quad n_5 | 6$

subgroup

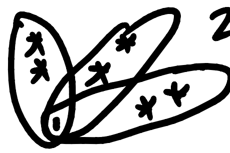
$n_2 \mid 15$ $\quad n_3 \mid 10$ $\quad n_5 \mid 6$ Dummit & Foote

$n_5 \equiv 1 \pmod{5}$

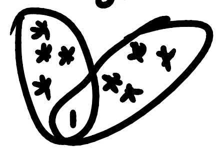
$n_2 \equiv 1 \pmod{2}$ $\quad n_3 \equiv 1 \pmod{3}$

$n_2 = 1, 3, 5, 15$ $\quad n_3 = 1, \underline{10}$ $\quad n_5 = 1, \underline{6}$

$n_3 = 10 \rightarrow$ 20 elements of order 3



$n_5 = 6 \rightarrow$ 24 elements of order 5



$n_3 = 1$ or $n_5 = 1$

$n_3 = 1 \Downarrow P_3$ normal

$n_5 = 1 \Downarrow P_5$ normal

P_3 normal $\Rightarrow G \xrightarrow{\pi} G/P_3$

$\pi^{-1}(P_5)$ has size 15

P_3 normal, P_5 normal $\Rightarrow$ $P_3 P_5$ is a subgroup

$a, c \in P_3$ normal

$b, d \in P_5$

$(ab)(cd)$

$= \underbrace{a\,b\,c\,b^{-1}}_{P_3}\,\underbrace{b\,d}_{P_5}$

in $H = P_3 P_5$, Sylow $n_3 = 1$ and $n_5 = 1$,

both normal $\Rightarrow H \cong C_3 \times C_5 \underset{\text{CRT using coprime}}{\cong} C_{15}$