Complex Analysis I

$f: \underset{\text{open}}{U} \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ differentiable if $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists (h is complex)

equivalently $\overset{=u+iv}{f}$ complex differentiable iff f real differentiable [as $\mathbb{R}^2 \to \mathbb{R}^2$] and satisfies $\underbrace{u_x = v_y \text{ and } u_y = -v_x}_{\text{Cauchy-Riemann equation}}$

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

TFAE:

- f is complex diff on all of U
- f is cont. complex diff ← holomorphic
- f is infinitely diff on U
- at each $z_0 \in U$, $f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$ in some nbhd of z_0 ← analytic

Morera's Theorem: f holo on U iff $\int_\Delta f(z)\,dz = 0$ for all (sufficiently small) triangles $\underline{\text{in } U}$ including interior of triangle

Contour integral: $\gamma: [a,b] \to U$

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt$$

← complex multiplication

If $f_n \to f$ unif, f_n analytic, then f analytic.

$$\int_\Delta f(z)\,dz = \int_\Delta \lim_{n\to\infty} f_n(z)\,dz \overset{*}{=} \lim_{n\to\infty} \underbrace{\int_\Delta f_n(z)\,dz}_{0}$$

$$\left| \int_\Delta f(z)\,dz - \int_\Delta f_n(z)\,dz \right|$$

$$= \left| \int_\Delta f(z) - f_n(z)\,dz \right|$$

$$\left| \int_a^b (f(\gamma(t)) - f_n(\gamma(t)))\gamma'(t)\,dt \right|$$

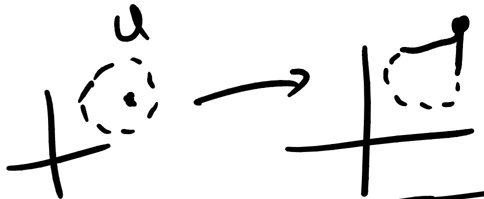
$$\leq \int^b M |\gamma'(t)|\,dt \qquad \leq \sup_\Delta |f(z) - f_n(z)| \cdot \underbrace{\text{length}(\Delta)}$$

$$\le \int_a^b M\,|\gamma'(t)|\,dt \qquad \le \underbrace{\sup_{\Delta} |f(z)-f_n(z)|}_{M} \cdot \underbrace{\text{length}(\Delta)}_{L}$$

Open mapping theorem

$f: \mathcal{U} \to \mathbb{C}$ nonconstant, then $f(\mathcal{U})$ open

$\Rightarrow$ Maximum modulus principal: If $|f(z)|$ attains a maximum in $\mathcal{U}$, then $f(z)$ constant



Ω Bound on $|f|$ on $\partial\Omega$ gives same bound Ω

Conformal: $f: \mathcal{U} \to \mathbb{C}$ injective

$\Rightarrow f: \mathcal{U} \to V$ bijection f is conformal map from $\mathcal{U}$ to V

$f'=0$ $\Rightarrow f' \neq 0$ on $\mathcal{U}$ $\mathcal{U}, V$ conformally equivalent

$f(z_0) + (z-z_0)^k g(z)$ $\Rightarrow f^{-1}: V \to \mathcal{U}$ holomorphic

$k \geq 2 \quad g(z_0) \neq 0$

$\Rightarrow f(z)$ angle preseving

Riemann Mapping Thm: If $\mathcal{U}$ simply connected (no holes), $\mathcal{U} \neq \mathbb{C}$, then $\mathcal{U}$ is conformally equivalent to $\mathbb{D}$.

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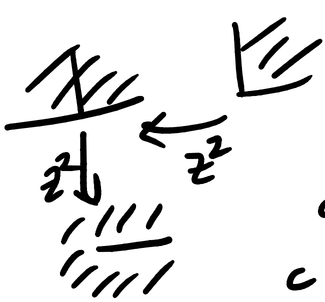
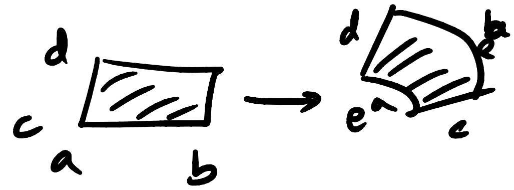
z^k, $\exp(z)$, $\dfrac{az+b}{cz+d}$ Möbius Transformations

Linear Fractional Transf...

$\sin(z), \cos(z)$ $az \quad z+b \quad \frac{1}{z}$ $ad-bc \neq 0$





$\hat{\mathbb{C}} \xrightarrow{\text{bij}} \hat{\mathbb{C}}$

$\mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$

circles/lines to circles/lines

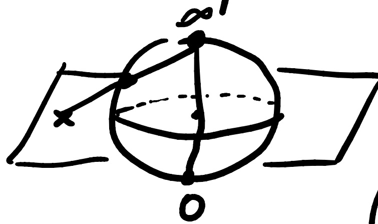
(lines are circles through ∞)

determined by where they send 3 points

$$\infty \mapsto \frac{a}{c}$$

$$z = \frac{-d}{c} \mapsto \infty$$

$$\left.\begin{array}{l} z_1 \mapsto w_1 \\ z_2 \mapsto w_2 \\ z_3 \mapsto w_3 \end{array}\right\} \longrightarrow \frac{az+b}{cz+d}$$



$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} < \mathbb{C}^2 \setminus \{0\} / \mathbb{C} \right.$$

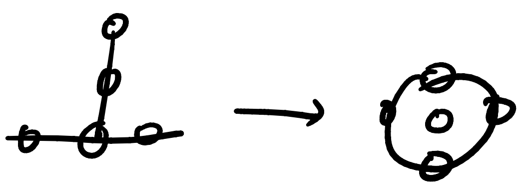
$$\frac{az+b}{cz+d} \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\frac{az+b}{cz+d} \circ \frac{ez+f}{gz+h} \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$\left(\frac{az+b}{cz+d}\right)^{-1} \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \sim \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

↑ as function

Ex: $\mathbb{H} \to \mathbb{D}$



$$\frac{z-i}{z+i}$$

$0 \mapsto -i$

$i \mapsto 0$

$\infty \mapsto i$

$\pm 1 \mapsto \pm 1$

$$i\,\frac{z-i}{z+i}$$

$|f(z)| \le 1$ for $|z| \le 1$
$|f(z)| < 1$ for $|z| < 1$

Schwarz lemma: $f: \mathbb{D} \to \mathbb{D}$, $f(0) = 0$,

$\Rightarrow |f(z)| \le |z|$, eq at any $z \ne 0 \Rightarrow f = e^{i\theta} z$

$|f'(z)|$

$\mathrm{Aut}(\mathbb{D}) = \{ e^{i\theta} \frac{z-a}{\ldots} \}$ $\quad$ $\mathrm{Aut}(\mathbb{C}) = \{az+b\}$

$$\mathrm{Aut}(\mathbb{D}) = \left\{ e^{i\theta} \frac{z-a}{1-\bar{a}z} \right\}, \quad \mathrm{Aut}(\mathbb{C}) = \{ az+b \}$$

(Conformal $\mathbb{D} \to \mathbb{D}$)

$$\mathrm{Aut}(\hat{\mathbb{C}}) = \left\{ \frac{az+b}{cz+d} \right\}$$

(more generally: any holo $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is rational)

Isolated Singularities

$f: U \setminus \{z_0\} \to \mathbb{C}$ holo

f has a Laurent expansion

$$\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

- removable ($a_k = 0$ for $k \leq -1$)
  equivalent to f bounded near z_0
  (Riemann's Thm)
- pole ($a_k = 0$ for sufficiently negative k)
  equivalent to $f \to \infty$ as $z \to z_0$
- essential singularity ($a_k \neq 0$ for ∞ly many $k \leq -1$)
  Casorati-Weierstass:
  Image $f(B_\varepsilon(z_0) \setminus \{z_0\})$ dense in $\mathbb{C}$

Liouville's Thm: $f: \mathbb{C} \to \mathbb{C}$ bounded, then constant