Rings I

Outline

See "Prelim Workshop Lectures Notes" for review of basic definitions (S85 14, F06 4A).

Fields $\subsetneq$ Euclidean Domains $\subsetneq$ PIDs $\subsetneq$ UFDs $\subsetneq$ integral domains (S03 9A, 6.10.5, 6.10.15).

- Using a Euclidean algorithm to determine gcd and remainder. Examples such as $\mathbb{Z}$, $k[x]$ where k a field, and the Gaussian integers.
- Prime and maximal ideals. Characterization by quotients. Prime ideal is maximal in PID.
- Local rings. A ring is local iff the set of non-units forms an ideal.
- An element is prime $\leftrightarrow$ irreducible in UFD. Only $\rightarrow$ in general integral domain.
- Quadratic integer rings.

Polynomials (F05 4A, 6.11.2, 6.11.9, 6.11.24, 6.11.28)

- Euclidean algorithm for $k[x]$ when k a field. In $R[x]$ for general R, can use division algorithm to divide by $f(x)$ when leading coefficient of f is a unit.
- Formulas for coefficients in terms of roots ("Vieta's formulas").
- Cyclotomic polynomials/irreducible factorization of $x^n - 1$.
- Rational root theorem.
- Gauss' lemma.
- Proving irreducibility of polynomials: Eisenstein's criterion. Show irreducibility over $\mathbb{Z}_p[x]$ for any prime p that does not divide leading coefficient. Substitute x for $x + \alpha$ for some α and consider the resulting polynomial. If degree 2 or 3: reducible polynomial must have a root in the underlying ring. If degree 4: assume factorization into quadratic polynomials with undetermined coefficients, show there is no solution (and also check no root).

Problems

Spring 1985 14 Let F be a field and let $M_n(F)$ be the ring of $n \times n$ matrices with coefficients in F. Prove that $M_n(F)$ has no nontrivial (two-sided) ideals. What can you conclude about ring homomorphisms from $M_n(F)$?

Spring 2003 9A Let R be the set of complex numbers of the form

$$a + 3bi, \ \ a, b \in \mathbb{Z}.$$

Prove that R is a subring of $\mathbb{C}$, and that R is an integral domain but not a unique factorization domain.

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Fall 2005 4A Let m and n be positive integers. Prove that the ideal generated by $x^m - 1$ and $x^n - 1$ in $\mathbb{Z}[x]$ is principal.

Fall 2006 4A Let R be a finite commutative ring with unity which has no zero-divisors and contains at least one element other than 0. Prove that R is a field.

6.10.5 Let F be a field and X a finite set. Let $R(X, F)$ be the ring of all functions from X to F, endowed with pointwise operations. What are the maximal ideals of $R(X, F)$?

6.10.15 Let R be a principal ideal domain and let I and J be nonzero ideals. Show that $IJ = I \cap J$ if and only if $I + J = R$.

6.11.2 By the fundamental theorem of algebra, the polynomial $x^3 + 2x^2 + 7x + 1$ has three complex roots, α_1, α_2 and α_3. Compute $\alpha_1^3 + \alpha_2^3 + \alpha_3^3$.

6.11.9 Let I be the ideal in $\mathbb{Z}[x]$ generated by 5 and $x^3 + x + 1$. Is I prime?

6.11.24 Let $f_n(x) = x^{n-1} + x^{n-2} + ... + x + 1$. Show that $f_n(x)$ is irreducible in $\mathbb{Q}[x]$ if n is prime. What if n is composite?

6.11.28 Factor $x^4 + x^3 + x + 3$ completely in $\mathbb{Z}_5[x]$.