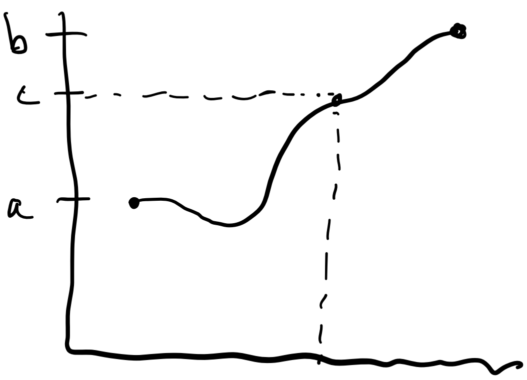
- Make sure to review derivatives and Taylor series of common functions.

$\mathbb{R}$ is the unique totally ordered field w/ the least upper bound property.

w/ continuity + differentiability when necessary give intermediate value theorem and mean value theorem.

IVT



MVT

A subset of $\mathbb{R}^n$ is compact $\iff$ it is closed and bounded (Heine-Borel). Also compactness $\iff$ sequential compactness.

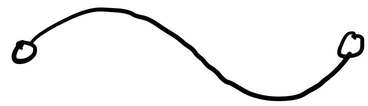
1.1.10 - Fall 1982 11

1. Two proofs

a) Image must be compact (it isn't)

b) Consider $x_n \in f^{-1}((0, 1/n))$. Then by compactness a subsequence converges to say x. Then by continuity $f(x) = 0$.

2. Take $\frac{1}{2} + \frac{\sin(2\pi x)}{2}$



3. Suppose there exists a continuous bijection

$$f : (0,1) \to [0,1]$$

then consider $x_0 \in f^{-1}(0)$, $x_1 \in f^{-1}(1)$. Then by the IVT every value is achieved between x_0 & x_1. Thus it can't be injective. ✓

1.5.3 Fall 1990 4 Note that

$$\int_0^1 x^2 dx = \frac{1}{3} x^3 \Big|_0^1 = 1/3$$

so

$$\frac{1}{3} f(\xi) = \int_0^1 f(\xi) x^2 dx$$

Thus we're looking for ξ s.t.

$$\int_0^1 (f(x) - f(\xi))\, x^2 dx = 0$$

Consider the continuous function $g: [0,1] \to \mathbb{R}$

$$\xi \longmapsto \int_0^1 (f(x) - f(\xi))\, x^2 dx$$

Let ξ_{min} and ξ_{max} be the extremizers of f.

Then

$$g(\xi_{min}) \geq 0$$
$$g(\xi_{max}) \leq 0$$

so done by intermediate value theorem. ✓

Sequences

A monotonic sequence $\{a_n\}$ converges $\Leftrightarrow$ it is bounded in which case

i) If increasing $a_n \to \sup\{a_n\}$
ii) If decreasing $a_n \to \inf\{a_n\}$

A sequence $\{a_n\} \subseteq \mathbb{R}$ may not converge to a finite number or $\pm\infty$, but there are two associated sequences which always do

i) $a_k^s = \sup\{a_n\}_{n \geq k}$

ii) $a_k^i = \inf\{a_n\}_{n \geq k}$

i) is decreasing, ii) is increasing, thus they always converge or diverge to $\pm\infty$. Denote them by

i) $\limsup a_n$

ii) $\liminf a_n$

Note $\limsup a_n \geq \liminf a_n$ always. Equality $\Leftrightarrow$ a_n converges in which case

$$\limsup a_n = \liminf a_n = \lim a_n$$

1.3.8 - Spring 2003 6A

Strategy is to show $\limsup x_n \leq X$, $\liminf x_n \geq X$ in which case these are equalities.

We write

$$\frac{x_n + (2x_{n+1} - x_n)}{2} = x_{n+1}$$

Take lim sup of both sides: Since $\sup(a_n + b_n) \le \sup a_n + \sup b_n$

$$\frac{\limsup x_n + x}{2} \ge \limsup x_n$$

Similarly

$$\frac{\liminf x_n + x}{2} \le \liminf x_n$$

So as long as $\limsup x_n$ and $\liminf x_n$ are finite we have $\limsup x_n \le x$, $\liminf x_n \ge x$ as desired.

We show $\{x_n\}$ must be bounded. Show by induction.

Once again

$$|x_{n+1}| \le \frac{1}{2}\left(|x_n| + |2x_{n+1} - x_n|\right)$$

can do since converges

Take M such that $M \ge \max\left(|x_1|, |2x_{n+1} - x_n|\right)$ for all n. Then $|x_1| \le M$ and assuming $|x_n| \le M$ then the above gives $|x_{n+1}| \le M$.

✓

<u>1.3.9. - Spring 2000 5</u> First suppose it converges, then

$$\lim x_n = \frac{1}{2}\left(\lim x_n + \frac{a}{\lim x_n}\right)$$

$$\Rightarrow \frac{1}{2}\lim x_n = \frac{1}{2}\frac{a}{\lim x_n} \Rightarrow (\lim x_n)^2 = a$$

$$\Rightarrow \lim x_n = \sqrt{a}$$

Also complete the square to see for any positive b

$$\frac{1}{2}\left(b + \frac{a}{b}\right) - \sqrt{a} = \frac{1}{2}\left(\sqrt{b} - \frac{\sqrt{a}}{\sqrt{b}}\right)^2 \geq 0$$

so $x_n \geq \sqrt{a}$. Further

$$x_n - x_{n-1} = \frac{1}{2}\left(\frac{a}{x_{n-1}} - x_{n-1}\right) \leq \frac{1}{2}\left(\sqrt{a} - x_{n-1}\right)$$

$$\leq 0$$

So the sequence is monotonically decreasing and bounded from below and thus converges to $\sqrt{a}$.

✓

Cauchy Schwarz + Taylor's thm

Consider the set of functions

$$\left\{ f : [a,b] \to \mathbb{C} \;\middle|\; \int_a^b |f|^2 dx < \infty \right\}$$

there is a natural inner product

$$\langle f, g \rangle = \int_a^b f \bar{g} \, dx$$

and thus as for any inner product $|\langle f,g \rangle| \leq \|f\| \, \|g\|$ called the Cauchy Schwarz inequality. In our case

$$\int_a^b f\bar{g} \, dx \leq \left(\int_a^b |f|^2 dx \right)^{1/2} \left(\int_a^b |g|^2 dx \right)^{1/2}$$

with equality $\iff$ $f = \lambda g$.

1.5.9 - Fall 1985 15 Apply Cauchy-Schwarz in a clever way:

$$a = \int_0^1 x f(x) dx = \int_0^1 \left(x \sqrt{f(x)} \right) \left(\sqrt{f(x)} \right) dx$$

$$\leq \left(\int_0^1 x^2 f(x) dx \right)^{1/2} \left(\int_0^1 f(x) dx \right)^{1/2}$$

$$= a$$

So Cauchy-Schwarz is an equality and thus

$$x\sqrt{f(x)} = c\sqrt{f(x)}$$

only possible if $f \equiv 0$, contradicting $\int_0^1 f(x)dx = 1$.

✓

If f is a $(k+1)$ differentiable function then its kth order Taylor polynomial centered at a is

$$f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \dots + \frac{1}{k!}f^{(k)}(a)(x-a)^k$$

The idea is this approximates f near a. Define the remainder $R_k(x)$ to be the difference of f and its kth order Taylor polynomial.

<u>Thm</u> (Taylor's thm) There is some $\xi \in [a,x]$ s.t.

$$R_k(x) = \frac{1}{(k+1)!}f^{(k+1)}(\xi)(x-a)^{k+1}$$

<u>1.1.17 - Fall 1997 15</u> Take any $x_0, x \in \mathbb{R}$. Then there is $\xi \in [x_0,x]$ s.t.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi)}{2}(x-x_0)^2$$

This allows us to write f' in terms of f and f''!

Then

$$f'(x_0)(x-x_0) = f(x) - f(x_0) - \frac{f''(\xi)}{2}(x-x_0)^2$$

$$\Rightarrow \qquad f'(x_0) = \frac{f(x)-f(x_0)}{(x-x_0)} - \frac{f''(\xi)}{2}(x-x_0)$$

thus

$$|f'(x_0)| \le \frac{2A}{(x-x_0)} + \frac{(x-x_0)B}{2}$$

Then set $(x-x_0) = 2\sqrt{\frac{A}{B}}$

✓

Multivariable Calc Crash Course

For a smooth function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ the <u>gradient</u> is the vector field

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right)$$

∇f is orthogonal to the level sets of f, $f^{-1}(k)$

A <u>critical point</u> of f is $p \in \mathbb{R}^n$ s.t. $\nabla f = 0$.

The <u>Hessian</u> is the $n \times n$ matrix

$$H(f) = \left(\frac{\partial f}{\partial x_i \partial x_j} \right)_{ij}$$

it is symmetric. If p is a critical point and the Hessian at p is positive (negative) definite, then $f(p)$ is a local maximum (minimum).

In the $n=2$ case, p a critical point

$$\det\left(H_p(f)\right) > 0 \Rightarrow \begin{array}{l} \text{local max if } f_{xx} > 0 \text{ or } f_{yy} > 0 \\ \text{local min if } f_{xx} < 0 \text{ or } f_{yy} < 0 \end{array}$$

$$\det\left(H_p(f)\right) < 0 \Rightarrow \text{Saddle point}$$

otherwise inconclusive.

For a smooth function $f: \mathbb{R}^n \to \mathbb{R}^m$ the Jacobian is the $m \times n$ matrix

$$\left(\frac{\partial f^i}{\partial x_j}\right)_{ij} \qquad (f^i \text{ is the } i\text{th coord})$$

<u>Thm</u> (Inverse function thm) If $f: \mathbb{R}^n \to \mathbb{R}^m$ smooth s.t. the jacobian of f at $p \in \mathbb{R}^n$ is invertible, then there are open sets $p \in U$, $f(p) \in V$ s.t.

$$f: U \to V$$

is smooth w/ smooth inverse.

Spring 1996 12 Just check the Jacobian is invertible at 0 and apply Inverse function thm.

$$F(X) = \mathrm{Id}(X) + G(X)$$

where $G(X) = X^2$. X^2 contains only degree 2 terms, thus all of the partial derivatives vanish at 0. Thus the Jacobian of F at 0 is the identity matrix.