Similarity

V a finite dimensional vector space

A basis $v_1, \ldots, v_n$ for V induces an isomorphism

$$V \longrightarrow F^n$$
$$v_i \longmapsto e_i$$

So, if $T: V \longrightarrow V$ a linear map, a basis $\mathcal{B}$ for V induces a linear map $T_{\mathcal{B}}: F^n \longrightarrow F^n$

If Z is a different basis, how does T_Z compare to $T_{\mathcal{B}}$? $\underline{A}$: There exists an isomorphism $P: F^n \longrightarrow F^n$ s.t.

$$T_Z = P^{-1} T_{\mathcal{B}} P$$

Defn Two square matrices A and B are called similar if there is an invertible matrix P s.t.

$$B = P^{-1} A P$$

Diagonalizable means similar to a diagonal matrix.

Rational Canonical Form Let R be a PID.

There is a classification of finitely generated R-modules
M:

Thm 1) $M \cong R^r \oplus R/_{(a_1)} \oplus \cdots \oplus R/_{(a_m)}$

$r \geq 0$, $a_1, \ldots, a_m \in R$ non units s.t. $a_1 | \ldots | a_m$

2) $r, a_1, \ldots, a_m$ unique up to units.

Now, R is a UFD, so can write $a \in R$ as

$$a = u \, p_1^{\alpha_1} \cdots p_s^{\gamma_s}$$

u a unit
$p_1, \ldots, p_s$ prime

Then by Chinese remainder theorem

$$R/_{(a)} \cong R/_{(p_1^{r_1})} \oplus \cdots \oplus R/_{(p_s^{r_s})}$$

Thm If M a fin gen R-module

$$M \cong R^r \oplus R/_{(p_1^{\alpha_1})} \oplus \cdots \oplus R/_{(p_s^{r_s})}$$

$r \geq 0$, $p_1^{\alpha_1}, \ldots, p_s^{r_s}$ powers of possibly nondistinct primes,
also unique up to units.

Now, if $T: V \to V$ a linear map, we can turn V into an $F[x]$-module by having x act by T i.e.

$$(a_n x^n + \dots + a_1 x + a_0) v$$
$$:= a_n T^n v + \dots + a_1 T v + a_0 v$$

V is finite dimensional so this is a finitely generated $F[x]$-module so

$$V = \frac{F[x]}{(a_1(x))} \oplus \frac{F[x]}{(a_2(x))} \dots \oplus \frac{F[x]}{(a_m(x))}$$

$$a_1(x) \mid \dots \mid a_m(x)$$

unique if require the a_i to be monic

Note: a_m is the minimal polynomial.

If $a(x) = x^k + b_{k-1} x^{k-1} + \dots + b_1 x + b_0$, then $1, \bar{x}, \dots, \bar{x}^{k-1}$ is a basis for the F-vector space $\frac{F[x]}{(a(x))}$.

On this basis multiplication acts by

$$1 \longmapsto \bar{x}$$
$$\bar{x} \longmapsto \bar{x}^2$$
$$\vdots$$

$$\bar{x}^{k-2} \longmapsto \bar{x}^{k-1}$$

$$\bar{x}^{k-1} \longmapsto -b_0 - b_1 \bar{x} - \cdots - b_{k-1}\bar{x}^{k-1}$$

which has matrix representation

$$C_a := \begin{pmatrix} 0 & 0 & - & - & - & - & -b_0 \\ 1 & 0 & & & & & -b_1 \\ 0 & 1 & & & & & \vdots \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & - & - & - & 1 & -b_{k-1} \end{pmatrix}$$

Thus there is a basis of V s.t. T becomes

$$\begin{pmatrix} C_{a_1} & & & \\ & C_{a_2} & & \\ & & \ddots & \\ & & & C_{a_m} \end{pmatrix}$$

This is the rational canonical form of T.

This is the canonical form of T. Two matrices over F are similar $\Longleftrightarrow$ they have the same rational canonical form.

Note the characteristic polynomial of C_a is $\pm a$, thus the characteristic polynomial of T is $\pm a_1 \cdots a_m$ which proves

Cayley Hamilton and that the characteristic poly and minimal polynomial have the same factors.

E.g. $A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}$ Char poly is $(x-2)^2(x-3)$

minimal poly is $(x-2)(x-3)$

Thus $a_1 = (x-2)$, $a_2 = (x-2)(x-3) = x^2 - 5x + 6$

so rational canonical form is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & 1 & 5 \end{pmatrix}$$

7.2.10 Let R_A, R_B the rational canonical forms of A and B respectively. Since A and B are real, the rational canonical forms over $\mathbb{C}$ are the same as the rational canonical forms over $\mathbb{R}$. Then $R_A = R_B$ since we assumed A, B similar over $\mathbb{C}$, thus they are also similar over $\mathbb{R}$. ✓

Jordan Canonical Form

Let $T : V \longrightarrow V$ linear. Assume minimal polynomial factors into linear terms (always true over $\mathbb{C}$). Then

factoring $a_1, \ldots, a_m$ gives

$$V \cong \frac{F[x]}{(x-\lambda_1)^{k_1}} \oplus \ldots \oplus \frac{F[x]}{(x-\lambda_\ell)^{k_\ell}}$$

λ_i may be repeated but they are each roots of the minimal polynomial (i.e. eigenvalues)

with respect to the basis $(\bar{x}-\lambda)^{k-1}, \ldots, \bar{x}-\lambda, 1$

x acts on $\dfrac{F[x]}{(x-\lambda)^k}$ as

$k \times k$
$$\begin{pmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & & \\ & & \ddots & \ddots & & \\ & & & & \lambda & 1 \\ & & & & & \lambda \end{pmatrix}$$

This is a <u>Jordan block of size k with eigenvalue λ</u>

Such a Jordan block has minimal polynomial $(x-\lambda)^k$.
It has only λ as an eigenvalue w/ dim of eigenspace 1.

T can be represented as a matrix

$$\begin{pmatrix} J_1 & & & \\ & \ddots & & O \\ & & \ddots & \\ O & & & J_\ell \end{pmatrix}$$

where J_i is a Jordan block of size k_i w/ eigenvalue λ_i. This is <u>Jordan canonical form</u> unique up to permutation. Note the minimal polynomial of T is the least common multiple of the $(x-\lambda_i)^{k_i}$

<u>7.6.24</u> The char poly of A is $-(x-1)^3$. The eigenvalues are just 1. The eigenvectors are x s.t.

$$(A-I)x = \begin{pmatrix} | & | & | \\ | & | & | \\ | & | & | \end{pmatrix} x = 0$$

These are vectors of the form

$$\begin{pmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\underbrace{\hspace{6cm}}$
basis for eigenspace

Minimal poly divides char poly so it's $(x-1), (x-1)^2$ or $(x-1)^3$.

$$(A-I)^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^2 = O$$

so minimal poly is $(x-1)^2$. There is then a Jordan block of size 2 so

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the Jordan canonical form.

7.6.30 Triangular so char poly is $(x-1)^6$.

Minimal poly is then a power of $(x-1)$. Direct calculation gives

$$(A-I)^2 \neq O, \quad (A-I)^3 = O$$

so minimal poly is $(x-1)^3$. Thus there is a 3x3 block

To determine the rest, find eigenspace of 1: Solve $Ax = x$

i.e.

$$x_1 = x_1$$
$$x_1 + x_i = x_i \quad 2 \leq i \leq 5$$
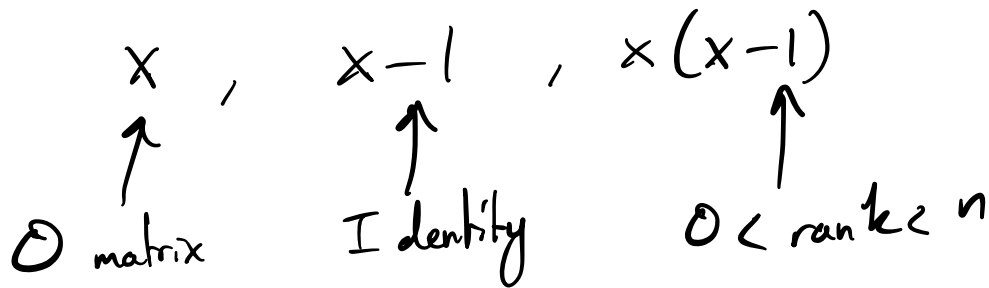$$x_1 + x_2 + \cdots + x_6 = x_6$$

$\Rightarrow x_1 = 0, \quad x_2 + x_3 + x_4 + x_5 = 0, \quad x_6$ undetermined

so 4 degrees of freedom. Thus 3 1×1 blocks

and 1 3×3 block, all eigenvalue 1 $\checkmark$

7.7.6 By assumption they both satisfy $x(x-1)$.

Possible minimal poly's are

$$x \qquad , \qquad x-1 \qquad , \qquad x(x-1)$$

$\uparrow$ $\qquad$ $\uparrow$ $\qquad$ $\uparrow$

$\mathbf{O}$ matrix $\qquad$ Identity $\qquad$ $0 < \text{rank} < n$

Since minimal poly is least common multiple of minimal polys of Jordan blocks, each Jordan block is 1×1. Then if they have the same rank they have the same Jordan canonical forms, so they're similar. $\checkmark$

The Jordan Canonical form gives

Thm $A \in M_{n \times n}(F)$ is diagonalizable $\iff m_A$ factors into linear terms in F w/ no repeated roots

Thm $A \in M_{n \times n}(F)$ triangularizable $\iff m_A$ (equivalently p_A) factors into linear terms over F.

All complex matrices are triangularizable and has an eigenvector.

7.5.7 $M_{n \times n}(\mathbb{C})$ is fin dim and matrices commuting with a fixed matrix $A \in M_{n \times n}(\mathbb{C})$ is a subspace. Thus can assume S is finite. Go by induction. Suppose $A_1, \ldots, A_n$ commuting and $A_1, \ldots, A_{n-1}$ have a common eigenvector. Let E be the space of all such common eigenvectors.

Let $v \in E$. Then $A_i A_{n+1} v = A_{n+1} A_i v = \lambda_i A_{n+1} v$ for all i so $A_{n+1} v \in E$. Then can view A_{n+1} as

a map
$$E \to E$$
since A_{n+1} is complex, it has an eigenvector in E. ✓

7.6.17 Changing basis we can write

T as a matrix

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

where A is the matrix of $T|_W : W \to W$
Want to show m_A factors into linear terms w/
none repeated. The matrix of T^k is
$$\begin{pmatrix} A^k & \underleftarrow{\text{some thing}} \\ 0 & B^k \end{pmatrix}$$

so any polynomial satisfied by T is also
satisfied by A. Thus $m_A \mid m_T$ and must

also factor into linear terms since M_T does. ✓

Inner / Hermitian Products

Last time we covered inner products on real vector spaces, the spectral theorem was if $T: V \to V$ a linear endomorphism of a real inner product space V, then

$$T^* = T$$

$$\Longleftrightarrow$$

T has an orthonormal basis of eigenvectors

$$\Longleftrightarrow$$

There is a matrix U s.t. $U^T = U^{-1}$ and diagonal $\longrightarrow$ $D = U^T A U$

where A is a representation of T in an orthonormal basis.

If V is a complex vector space then there is no bilinear (over $\mathbb{C}$), symmetric, positive definite form on V. Instead define <u>Hermitian inner product</u>

$$\langle , \rangle : V \times V \longrightarrow \mathbb{C}$$

to be s.t.

i) $\langle a v_1 + b v_2, w \rangle = a \langle v_1, w \rangle + \bar{b} \langle v_2, w \rangle$

$\langle v, a w_1 + b w_2 \rangle = \bar{a} \langle v, w_1 \rangle + \bar{b} \langle v, w_2 \rangle$

ii) $\langle v, w \rangle = \overline{\langle w, v \rangle}$

iii) $\langle v, v \rangle \geq 0$ equality $\iff v = 0$

$\mathbb{C}^n$ has a standard hermitian inner product

$$\langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n$$

Then if $T : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ linear, its __Hermitian adjoint__

is $T^* : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ defined by

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

if A is the matrix of T, then the matrix of

T^* is $\bar{A}^T =: A^*$

Thm (Spectral thm over $\mathbb{C}$) $T: \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is

__normal__ i.e. $TT^* = T^*T \iff T$ has

an orthonormal basis of eigenvectors.

Note that __Hermitian__ maps $\left(\begin{array}{l} \text{i.e. } T \text{ s.t. } T = \overline{T}^* \\ \text{complex version of symmetric} \end{array} \right)$

are normal.

Gram-Schmidt: From a basis $v_1, \ldots, v_n$

we can obtain an orthonormal basis $e_1, \ldots, e_n$

by the procedure

$$e_1 = \frac{v_1}{\|v_1\|}$$

$$e_2 = \frac{v_2 - \langle v_2, e_1 \rangle}{\|v_2 - \langle v_2, e_1 \rangle\|}$$

$$\vdots$$

$$e_n = \frac{v_n - \langle v_n, e_1 \rangle - \ldots - \langle v_n, e_{n-1} \rangle}{\|v_n - \langle v_n, e_1 \rangle - \ldots - \langle v_n, e_{n-1} \rangle\|}$$