Studying linear maps $T: V \to W$, V, W vector spaces over a field F.

A set $v_1, \dots, v_n \in V$ is <u>linearly independent</u> if whenever $c_1, \dots, c_n \in F$ s.t.

$$c_1 v_1 + \dots + c_n v_n = 0 \implies c_1 = \dots = c_n = 0$$

i) A <u>basis</u> is a linear independent spanning set

ii) <u>Dimension</u> is the length of any basis.

If $T: V \to W$ linear then

$$\dim V = \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T))$$

when $\dim V = \dim W \implies$ injectivity $\iff$ surjectivity $\iff$ invertible

<u>2.1.1</u> Evaluating at a point is a linear map

$$P_3 \longrightarrow F$$

1. This is the kernel of

$$\underset{\dim 4 \to}{} P_3 \longrightarrow F$$
$$p \longmapsto p(1)$$

dim of kernel is $4 - 1 = 3$ so 4 polynomials in the kernel are automatically linearly dependent.

2. Not a kernel, $\{1, 1+x, 1+x^2, 1+x^3\}$ a counter example.

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The λ-eigenspace of a linear map $T: V \to V$ is

$$\mathrm{Ker}(T - \lambda I)$$

T is diagonalizable if there is a basis $v_1, \dots, v_n$ of V s.t. v_i is an eigenvector for T.

For any linear map $T: F^n \to F^m$ there is an $m \times n$ matrix A s.t. $T(x) = Ax$ for all $x \in F^n$.

A is given by

$$\left(T(e_1) \middle| \cdots \middle| T(e_n) \right) \qquad e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th place}$$

Diagonalizable means we can choose a basis for which A is diagonal. Makes it easier to compute powers of T.

An inner product on a real vector space V is a bilinear, symmetric map $\langle\, , \,\rangle : V \times V \longrightarrow \mathbb{R}$ that is positive definite (that is, $\langle v, v\rangle \geq 0$, $= 0 \Leftrightarrow v = 0$)

If $T: V \to W$ is linear between inner product spaces, the <u>adjoint</u> is

$$T^*: W \longrightarrow V$$

$$\langle Tv, w\rangle = \langle v, T^* w\rangle \text{ for all } \begin{matrix} v \in V \\ w \in W \end{matrix}$$

If $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ has matrix A, the matrix of T^* is A^T.

<u>Thm</u> (Spectral thm) $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ has an orthonormal basis of eigenvectors $\Leftrightarrow$ $T = T^*$

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"symmetric" or "self-adjoint".

<u>7.2.7.</u> T is symmetric thus diagonalizable. Note $2 \Rightarrow 1$ so we show each eigenspace has dimension at most 1.

We solve the system of equations $Tx = \lambda x$. First equation is

$$a_1 x_1 + b_1 x_2 = \lambda x_1$$

$b_1 \neq 0$ so can solve for x_2 in terms of x_1.

Suppose we can solve $x_1, \dots, x_i$ in terms of x_1. Then i^{th} eqn is

$$b_{i-1} x_{i-1} + a_i x_i + b_i x_{i+1} = \lambda x_i$$

since $b_i \neq 0$ can solve for x_{i+1} in terms of x_1.

Last equation is

$$b_{n-1} x_{n-1} + a_n x_n = \lambda x_n$$

If we can solve for x_n, then λ is an eigenvalue and eigenspace only depends on one parameter, thus 1-dim. Otherwise not an eigenvalue.

<u>Determinant</u>

Determinant of a square matrix computed as follows:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

then for $A = (a_{ij})$, let A_{ij} be the submatrix w/ ith row and jth column deleted. Fix a row i, then

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Doesn't matter which row we choose.

E.g. $\det\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 2 \end{pmatrix} = \det\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} - \det\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

Properties of the determinant:

i) Multilinear, alternating in the columns

ii) $\det(AB) = \det(A)\det(B)$

iii) $\det A \neq 0 \iff A$ invertible

iv) $\det A = \det A^T$

7.2.11 A is called an $n \times n$ Vandermonde matrix.

By induction. Holds for

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}$$

Then in $n \times n$ case, let A_i denote A with the ith column and last row deleted. Then

$$\det A = \pm\left((\det A_n)x_n^{n-1} + \dots + (\det A_2)x_n + \det A_1\right)$$

$(n-1)$ degree poly in x_n, has at most $(n-1)$ roots.

Setting $x_n = x_i$, we see $\det A = 0$. Thus

$$\det A = c\prod_{i=1}^{n-1}(x_n - x_i)$$

$c = \det A_n$, A_n an $(n-1)\times(n-1)$ vandermonde so the result follows from induction.

7.2.12 1. If T not invertible, there is $x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \neq 0$

s.t.

$$Tx = 0$$

thus

$$x_n a_i^n + \dots + x_1 a_i + x_0 = 0$$

for each i. Then each a_i is a root of a degree n polynomial, only possible if not distinct.

2. Just invert T.

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Note if A is triangular, the determinant is the product of the diagonal.

Characteristic and minimal polynomial

Let A be an $n \times n$ matrix with entries in a field F. Then A satisfies a polynomial

$$p(x) = a_n x^n + \dots + a_1 x + a_0 \in F[x]$$

if

$$p(A) = a_n A^n + \dots + a_1 A + a_0 I = 0$$

$\{ p \in F[x] \mid p(A) = 0 \}$ is an ideal in $F[x]$. $F[x]$ is a pid so there is a unique monic polynomial $m_A \in F[x]$ that generates the ideal, called the minimal polynomial of A.

The characteristic polynomial of A is

$$F[x] \ni p_A = \det(A - xI)$$

so λ is an eigenvalue of $A \iff p_A(\lambda) = 0$

Important facts:

i) (Cayley-Hamilton) $m_A \mid p_A$, so in particular

$$p_A(A) = 0$$

ii) p_A divides some power of m_A, so p_A and m_A have same irreducible factors up to powers.
(so λ an eigenvalue $\iff m_A(\lambda) = 0$).

iii) If A is $n \times n$, p_A is degree n. Trace(A) is $n-1$ coeff of p_A, $\det A$ is the constant term.

7.5.3 $A^m = 0$ thus $m_A \mid x^m$, so $m_A = x^k$ for some $k \leq n$. Thus $A^n = 0$.

7.6.5 The characteristic polynomial of A is

$$x^3 - 8x^2 + 20x - 16 = (x-4)(x-2)^2$$

thus the minimal polynomial is either $(x-4)(x-2)^2$ or $(x-4)(x-2)$. Can check its $(x-4)(x-2)$.

Now, using euclidean algorithm

$(*)\ x^{10} = p(x)\, m_A(x) + ax + b \qquad a, b \in \mathbb{R}$

Thus $A^{10} = p(A)\, m_A(A) + aA + bI = aA + bI$

Plugging $x = 2$ and $x = 4$ into $(*)$ we get

$$2^{10} = 2a + b$$
$$4^{10} = 4a + b$$

solve for a and b.

<u>Bonus determinant calculation</u>

Find determinant of an $n \times n$ matrix of the form:

$$\begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & \vdots \\ b & b & a & \ddots & b \\ \vdots & \vdots & \ddots & \ddots & b \\ b & b & \cdots & b & a \end{pmatrix}$$

Adding a row/column to another row/column does not change det so add all columns to first column to get

$$\begin{pmatrix} a+(n-1)b & b & b & \cdots & b \\ a+(n-1)b & a & b & \ddots & \vdots \\ \vdots & b & a & \ddots & \vdots \\ \vdots & b & \ddots & \ddots & b \\ \vdots & \vdots & \ddots & b & a \\ a+(n-1)b & b & \cdots & b & a \end{pmatrix}$$

then subtract the first row from the others to get

$$\begin{pmatrix} a+(n-1)b & b & \cdots & b & b & b \\ & a-b & 0 & \cdots & & 0 \\ & & a-b & 0 & \cdots & 0 \\ & 0 & & \ddots & & \vdots \\ & & & & & a-b \end{pmatrix}$$

so upper triangular, so determinant is

$$(a-b)^{n-1}(a+(n-1)b)$$