Groups

Wednesday, August 7, 2024

Subgroups $H \le G$

Index $[G:H]$ = number of cosets gH in G $= \dfrac{|G|}{|H|}$

$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ $\quad ij = k$
(quaternion)

Subgroups

$$\begin{array}{ccc} & Q_8 & \\ \langle i \rangle & \langle j \rangle & \langle k \rangle \\ & \{\pm 1\} & \\ & 1 & \end{array}$$

$\{\pm 1\} \leftarrow$ central (commutes with the whole group)

D_n (or D_{2n}) (dihedral)

group of symmetries of an n-gon

$n=3$ $\quad$ $n=4$

n rotations (subgroup of index 2)
n reflections $\quad C_n \rtimes C_2$

C_n (cyclic) $\cong \mathbb{Z}/n\mathbb{Z} = \langle 1 \rangle$

generated by 1 element, can be any $1 \le k \le n$ coprime to n
$\varphi(n)$ elements that each generate
$\mathrm{Aut}(C_n) \cong (\mathbb{Z}/n\mathbb{Z})^\times$
If $C_n = \langle g \rangle$, then $g \mapsto g^k$ for $\gcd(k,n)=1$ gives an automorphism

Symmetric Group S_n = permutations of $\{1, ..., n\}$

sign: $S_n \longrightarrow \{\pm 1\}$ $\qquad$ E.g., $1 \to 4 \to 3 \to 1$, $\; 2 \rightleftarrows 5$ permutation in S_5

$\mathrm{sign}(\sigma) = 1 \longleftrightarrow$ even permutation $\qquad (143)(25)$ cycle notation
$\mathrm{sign}(\sigma) = -1 \longleftrightarrow$ odd permutation

kernel = {even permutations} $= A_n$ (alternating group)

$[S_n : A_n] = 2.$ $\quad A_n$ is simple (no normal subgroups) for $n \ge 5$.

Matrix groups $\quad GL(3,5) = GL_3(\mathbb{F}_5)$ = invertible 3×3 matrices$/\mathbb{F}_5$
$SL(3,5) = SL_3(\mathbb{F}_5)$ = det 1 3×3 matrices$/\mathbb{F}_5$

Group Actions

$G \times X \to X$ $\qquad 1 \cdot x = x$, $\quad g \cdot (h \cdot x) = (gh) \cdot x$

$\mathrm{Stab}(x) = \{g \in G : g \cdot x = x\}$ subgroup of G

$\mathrm{Orb}(x) = \{g \cdot x : g \in G\}$

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Orbit-Stabilizer Theorem

$$|\mathrm{Orb}(x)| = [G : \mathrm{Stab}(x)] = \frac{|G|}{|\mathrm{Stab}(x)|}, \qquad \sum_{\substack{\text{one } x \\ \text{from each} \\ \text{orbit}}}' |\mathrm{Orb}(x)| = |X|$$

Conjugation action

$G \circlearrowright G \quad (G \times G \to G)$

$g \cdot h = ghg^{-1}$

$$\mathrm{Stab}(g) = \{h \in H : h^{-1}gh = g\} = \{h \in H : gh = hg\} = C_G(g)$$

$\mathrm{Orb}(g) = \{h^{-1}gh\}$ = conjugacy class of g

$$|\mathrm{conj\,class}(g)| = [G : C_G(g)] = \frac{|G|}{|C_G(g)|}$$

$$\sum_g^{*} |\mathrm{conjcl}(g)| = |G| \quad \text{(class equation)}$$

center $\{g : gh = hg, \forall h\}$

$$|Z(G)| + \sum_g^{**} \frac{|G|}{|C_G(g)|} = |G|$$

G finite group, $X = \{(g,h) : gh = hg\}$

a) Show $|X| = c|G|$, $c = \#$ conj classes

$$|X| = \sum_g |C_G(g)| = \sum_g \frac{|G|}{|\mathrm{conjclass}(g)|}$$

$$= |G| \sum_g \frac{1}{|\mathrm{conjclass}(g)|}$$

Each conjugacy class contributes 1 to the sum

$$= |G| \cdot c$$

b) Find $|X|$ for $G = S_5$.

Thm: Two permutations are conjugate in S_n $\Longleftrightarrow$ they have the same cycle.

$$\begin{bmatrix} 1+1+1+1+1 & 2+2+1 \\ 2+1+1+1 & 2+3 \\ 3+1+1 & \\ 4+1 & \\ 5 & \end{bmatrix}$$

$c = 7$

$|X| = 7 \cdot 5! = 7 \cdot 120 = 840.$

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c) Theorem $|G| = p^k m$, $p \nmid m$

L 5

Sylow's Theorems. $|G| = p^k m$, $p \nmid m$

A sylow p-subgroup is a subgroup of order p^k

I) Sylow p-subgroups exist (at least one)

II) Any two Sylow p-subgroups are conjugate

III) $n_p = \#$ Sylow p-subgroups $\quad n_p \equiv 1 \pmod{p}$

$n_p \mid |G|$ (actually $n_p \mid m$)

$$n_p = [G : N_G(P)] = |G| / |N_G(P)|, \quad N_G(P) = \{g : gPg^{-1} = P\}$$

(orbit-stabilizer theorem)

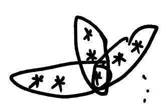
$n_p = 1 \iff P$ is normal

Section 6.2 of Dummit & Foote

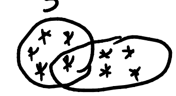
Show that if $|G| = 30$, then G has a cyclic subgroup of order 15.

n_2	n_3	n_5
$n_2 \mid 15$	$n_3 \mid 10$	$n_5 \mid 6$
$n_2 \equiv 1 \pmod{2}$	$n_3 \equiv 1 \pmod{3}$	$n_5 \equiv 1 \pmod{6}$
$n_2 = 1, 3, 5, 15$	$n_3 = 1, 10$	$n_5 = 1, 6$

We can rule out $n_3 = 10$ and $n_5 = 6$ occuring together



20 elements of order 3



24 elements of order 5

impossible since $20 + 24 > 30$

If $n_3 = 1$, then $P_3 \trianglelefteq G$ (normal)

$|\underbrace{G/P_3}| = 10$

Sylow $\Rightarrow$ has a subgroup of order 5

$|H/P_3| = 5 \Rightarrow |H| = 15$

If $n_5 = 1$, then $P_5 \trianglelefteq G$ (normal)

$|\underbrace{G/P_5}| = 6$

Sylow $\Rightarrow$ has a subgroup of order 3.

$|H/P_5| = 3 \Rightarrow |H| = 15$

So G has a subgroup H of order 15.

In H, $n_3 = 1$, $n_5 = 1$, so H

So G has a subgroup H of order 15.

In H, $n_3 = 1$, $n_5 = 1$, so H has normal subgroups $P_3, P_5 \trianglelefteq H$

☆ [Recognition theorem: If $H, K \trianglelefteq G$, $H \cap K = 1$, $|H| \cdot |K| = G$, then $G \cong H \times K$.

So $H \cong P_3 \times P_5 \cong \underbrace{\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}}_{\text{coprime}} \underset{\text{CRT}}{\cong} \mathbb{Z}/15\mathbb{Z}$ cyclic.

$H \leq G$ finite index $[G:H] = n$. Show G contains a normal subgroup $N \trianglelefteq G$, $N \leq H$, $[G:N] \leq n!$

(One application: If $[G:H] = 2$, then $H \trianglelefteq G$)

$G \circlearrowright$ cosets of H by $g \cdot (g'H) = (gg')H$

$\varphi : G \to S_n$ a homorphism

Set $N = \ker \varphi = \{ g \in G : gg'H = g'H, \ \forall g'H \}$

$= \bigcap_{g \in G} gHg^{-1}$ $\qquad$ $g'^{-1} g g' \in H \quad \forall g'$

$\qquad$ $g \in g' H g'^{-1} \quad \forall g'$

$N \trianglelefteq G$ since N is a kernel

$N \leq H$

$[G : \ker \varphi] = |\operatorname{im} \varphi| \,\big|\, |S_n|$ (so $\leq n!$)

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1) If G is nonabelian, then $G/Z(G)$ is not cyclic

Pf: If $G/Z(G) = \langle gZ(G) \rangle$

then any $hZ(G) = g^k Z(G)$

so $h = g^k \cdot z$ for some $z \in Z(G)$

But $(g^j \cdot z)(g^k \cdot z') = (g^k \cdot z')(g^j \cdot z)$

contradicts G nonabelian.

2) If $|G| = p^n$, show $|Z(G)| > 1$.

$\boxed{|G|} = |Z(G)| + \boxed{\sum_g \underbrace{\frac{|G|}{|C_G(g)|}}_{p^k}}$

div by p (if $n \geq 1$)

p^k = power of p, larger than 1

p (if $n \geq 1$)

So $p \mid |Z(G)|$

p^n = power ... larger than 1

div by p

3) If $|G| = p^2$, then

$|Z(G)| = 1$ or $|Z(G)| = p$ or $\underbrace{|Z(G)| = p^2}_{G \text{ is abelian}}$

impossible by (2)

impossible since then $|G/Z(G)| = p$ cyclic, contradicting (1)