Group Theory

Outline

Basics

- Checking group axioms
- Working with generators
- Key examples (quaternion, dihedral, cyclic, symmetric, alternating; major matrix groups)
- Isomorphism theorems

Special properties of cyclic groups (F00 16)

- Characterization by subgroups
- Automorphism groups

Special properties of the symmetric group (F05 8A)

- Transposition decompositions, the alternating group
- Disjoint cycle decompositions, conjugacy
- Generating subsets (adjacent transpositions, n-cycle and one transposition)

Finitely-generated abelian groups

- Classification (two descriptions of torsion part)

Direct and semidirect products (F04 9B)

- Construction of semidirect products
- Recognition theorems

Group actions (F03 7B, S08 3B)

- Orbit-stabilizer theorem - useful for many counting problems
- Class equation
- G acts on itself or a collection of its subgroups by conjugation. G acts on the cosets of a subgroup by left (or right) multiplication.
- Think of as a map into S_n. Look at image and kernel.

Sylow theorems ($\star$)

- Argue by size (be careful about sizes of intersections)
- Have G act on its Sylow subgroups by conjugation
- Cauchy theorem

Problems

Fall 2000 16 (Half of "characterization by subgroups") Let G be a finite group of order n with the property that for each divisor d of n there is at most one subgroup in G of order d. Show G is cyclic.

Fall 2003 7B (a) Let G be a finite group and let X be the set of pairs of commuting elements of G

$$X = \{(g, h) \subseteq G \times G : \; gh = hg\}.$$

Prove that $|X| = c|G|$ where c is the number of conjugacy classes in G.
(b) Compute the number of pairs of commuting permutations on five letters.

Fall 2004 9B Prove that every group of order 30 has a cyclic subgroup of order 15.

Fall 2005 8A Find the smallest n for which the permutation group S_n contains a cyclic subgroup of order 111.

Spring 2008 3B ("Poincare's Theorem") Let G be a group and $H \leq G$ a subgroup of finite index n. Show that G contains a normal subgroup N such that $N \leq H$ and the index of N is $\leq n!$.

Spring 2009 8B 1. Let G be a non-abelian finite group. Show that $G/Z(G)$ is not cyclic, where $Z(G)$ is the center of G.
2. If $|G| = p^n$, with p prime and $n > 0$, show that $Z(G)$ is not trivial.
3. If $|G| = p^2$, show that G is abelian.

($\star$) Use the simplicity of A_6 to show that A_6 does not have an index 3 subgroup. Then show that there are no simple groups of order 120.