Explicit formulas for some DEs

A linear ODE is of the form

$$a_n(x) f^{(n)}(x) + \dots + a_1(x) f'(x) + a_0(x) f(x) = g(x)$$

a_i, g are given. We may also demand f and its derivatives satisfy initial conditions (e.g. $f(0) = 0$, $f'(1) = 2$).

Named linear since

$$f \mapsto a_n(x) f^{(n)} + \dots + a_1(x) f' + a_0(x) f$$

is a linear map. If $g \equiv 0$ the ODE is called homogeneous.

Since the kernel is a subspace, homogeneous equations have the superposition property i.e. if f_1 and f_2 are solutions then so is

$$c_1 f_1 + c_2 f_2 \quad \text{for all } c_1, c_2 \in \mathbb{R}$$

A solution to an inhomogeneous equation (i.e. $g \not\equiv 0$) is called a particular solution. If f_1 and f_2 are particular solutions, then $f_1 - f_2$ is a solution to the homogenous equation. Thus to solve:

i) Find all homogenous solutions f_h (usually a superposition of finitely many)

ii) Find one particular solution f_p

iii) The general solution is then

$$f_p + f_h$$

iv) Plug in initial conditions to general soln.

Spring 2008 4A First solve the homogenous equation

$$y'' - 2y' - y = 0$$

$r^2 - 2r - r$ has roots $1 \pm \sqrt{2}$, thus

$$c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x}$$

are homogenous solutions. For a particular soln, $\boxed{\text{guess}}$ Ae^{-x}. Then

$$Ae^{-x} + 2Ae^{-x} - Ae^{-x} = e^{-x}$$

$\Rightarrow$ need $A = \frac{1}{2}$, so $\frac{1}{2}e^{-x}$ is a particular soln,

so general soln is

$$\frac{1}{2}e^{-x} + C_1 e^{(1+\sqrt{2})x} + C_2 e^{(1-\sqrt{2})x}$$

The initial condition gives

$$\frac{1}{2} + C_1 + C_2 = 0$$

$$-\frac{1}{2} + (1+\sqrt{2})C_1 + (1-\sqrt{2})C_2 = 0$$

which has a unique solution. ✓

A <u>linear system of m ODEs</u> is an equation of the form

$$x'(t) = A(t)x(t) + b(t)$$

where $A(t)$ is $m \times n$ matrix valued, b is $\mathbb{R}^n$-valued, and $x(t)$ is an unknown $\mathbb{R}^n$-valued function. If $b \equiv 0$, it is homogeneous. Steps i) – iv) for solving linear ODEs still apply. E.g. If A is a constant, diagonalizable matrix, the homogeneous solns are

$$C_1 e^{\lambda_1 t} v_1 + \ldots + C_n e^{\lambda_n t} v_n$$

where v_i is a λ_i-eigenvector.

Fall 2021 6A This is a symmetric matrix, making it easy to diagonalize, we will discuss later. For now,

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \text{ basis of the kernel}$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ basis of the 14 eigen space.}$$

General soln is then

$$y(t) = c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{14t} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Initial condition gives

$$-2c_1 - 3c_2 + c_3 = 1$$

$$c_1 + 2c_3 = 0$$

$$c_2 + 3c_3 = 0$$

So $c_1 = -2c_3$, $c_2 = -3c_3$, so first eqn becomes

$$14c_3 = 1 \Rightarrow c_3 = 1/14$$

✓

A couple more techniques. Consider a diff eq of the form

$$f'(x) = g(x)\,h(f(x))$$

if we sub $y = f(x)$, then

$$\frac{dy}{dx} = g(x)\,h(y)$$

$$\Rightarrow \frac{1}{h(y)}\,dy = g(x)\,dx$$

Then integrate both sides. This is called "separation of variables".

<u>Ex.</u>

$$\frac{dy}{dx} = xy \quad \Rightarrow \quad \frac{1}{y}\,dy = x\,dx$$

$$\Rightarrow \ln|y| = \frac{x^2}{2} + C$$

$$\Rightarrow y = k e^{x^2/2} \checkmark$$

Another technique is integrating factors. For an eqn of the form

$$f' + p(x) f = q(x)$$

Can take $r(x) = \int p$ so then

$$(e^r f)' = e^r(f' + pf)$$
$$= e^r q$$

Integrating both sides gives

$$f = Ce^{-r} + e^{-r}\int e^r q$$

<u>Fall 2021 1A</u>

a) $y_1(t) = t$ works

b) If $y(t)$ is another solution, then $u(t) = y(t) - y_1(t)$ is s.t.:

$$u' = y' - 1$$
$$= y^2 - ty$$
$$= (y-t)^2 + t(y-t)$$
$$= u^2 + tu$$

So u solves $u' = u^2 + tu \iff \frac{u'}{u^2} - \frac{t}{u} = 1$

Change variables to $z = \frac{-1}{u}$, so $z' = \frac{u'}{u^2}$ and

$$z' + tz = 1$$

Now apply the integrating factor $e^{\int t} = e^{t^2/2}$

$$\Rightarrow z(t) = Ce^{-t^2/2} + e^{-t^2/2}\int e^{t^2/2}$$

$$\Rightarrow u(t) = \frac{1}{Ce^{-t^2/2} - e^{-t^2/2}\int e^{t^2/2}}$$

✓

Establishing
Existence + Uniqueness

The following guarantees local existence + uniqueness under mild assumption.

Thm (Picard–Lindelöf) Suppose $f(t,y)$ is uniformly Lipschitz in y and continuous in t. Then for some $\varepsilon > 0$ the initial value problem

$$\frac{dy}{dt} = f(t, y(t)), \quad y(0) = y_0$$

has a unique solution on $(-\varepsilon, \varepsilon)$.

3.1.3 Fall 1993 14 This is a question of existence for the IVP

$$\frac{dy}{dx} = y^n$$

First separate variables: $\frac{1}{y^n} dy = dx$

$$\Rightarrow \frac{1}{(1-n)\, y^{n-1}} = x + C$$

$$\Rightarrow \frac{1}{y^{n-1}} = (1-n)x + C$$

$$C = \frac{1}{y(0)^{n-1}} > 0$$

So $y = \dfrac{1}{\left(C-(n-1)x\right)^{1/n-1}}$ solves the equation in $\left[0, \frac{C}{n-1}\right)$, it blows up as $x \mapsto \frac{C}{n-1}$ but is unique by Picard-Lindelöf, thus none defined on $[0, \infty)$ exist. ✓

Another tool is the implicit and inverse function theorems. Note that the derivative of an inverse function is given by $\left(f^{-1}(y)\right)' = \dfrac{1}{f'(f^{-1}(y))}$.

3.1.0 - Fall 1982

1. f is not necessarily Lipschitz so can't apply P-L. Instead, since f is nonvanishing we can invert y and x so instead consider

$$\frac{dx}{dy} = \frac{1}{f(y)}, \quad x(c) = 0$$

which is continuous in y, Lipschitz in x (no x dependence).
Now apply P-L, then invert back to y.

2. We need the solution to

$$\frac{dx}{dy} = \frac{1}{f(y)}, \quad x(c) = 0$$

to go to $\pm\infty$ to the right, $\mp\infty$ to the left.
So need

$$x(\infty) = \int_c^\infty dx = \int_c^\infty \frac{1}{f(y)}\,dy$$

$$x(-\infty) = \int_{-\infty}^c dx = \int_{-\infty}^c \frac{1}{f(y)}\,dy$$

diverge

✓

Gronwall's Inequality

This about extracting information about a function from a differential inequality.

Summer 1982 6 (Also on 2021 exam)

Consider $(e^{-x}f)' = e^{-x}f' - e^{-x}f = e^{-x}(f'-f) > 0$

$\Rightarrow$ $e^{-x}f$ strictly increasing. $e^{-x}f$ is zero at x_0,

so $e^{-x}f > 0$ for all $x > x_0$ $\Rightarrow$ $f > 0$ for all $x > x_0$

$\checkmark$

By generalizing this argument, one gets

<u>Thm</u> If $\eta : [0,T] \to \mathbb{R}$ differentiable, ϕ integrable

s.t.

$$\eta'(t) \le \phi(t)\,\eta(t)$$

Then

$$\eta(t) \le e^{\int_0^t \phi(s)\,ds}\,\eta(0)$$

pf differentiate $\left(\eta(t)\,e^{-\int_0^t \phi(s)\,ds}\right)$ and apply

assumptions

$\square$