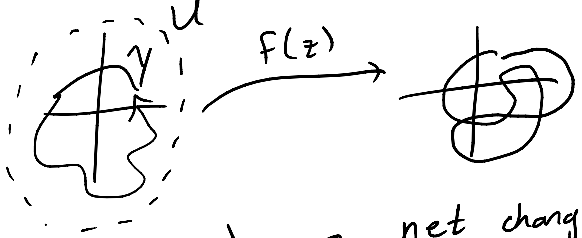
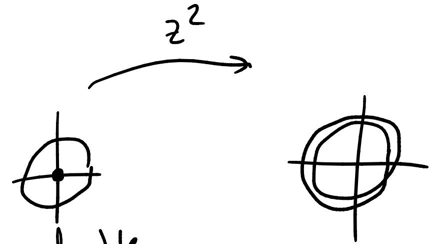
Argument Principle

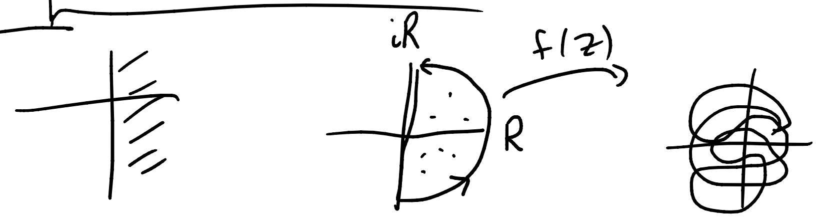


$2\pi(\#\text{zeros} - \#\text{poles})$ = net change in argument



$2\cdot(2\pi) = 4\pi$

α, β real nonzero



$f(z) = z^{2n} + \alpha^2 z^{2n-1} + \beta^2 = 0$

count # roots in the RHP $= \{z : \operatorname{Re}(z) > 0\}$ $-iR$

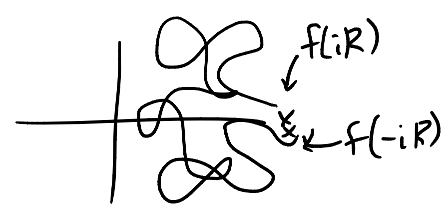
when $|z| = R$, z^{2n} dominates, so net change in argument is approximately $\underbrace{2n}_{\text{power}}\pi$ ← input change in argument

$f(it) = (-1)^n t^{2n} + \alpha^2 i (-1)^{n-1} t^{2n-1} + \beta^2$

n even: $f(iR), f(-iR)$

$\operatorname{Re}(f(it)) = t^{2n} + \beta^2 > 0$

total: $n \cdot 2\pi \Rightarrow$ exactly n zeros.



never winds around 0
net change in argument ≈ 0

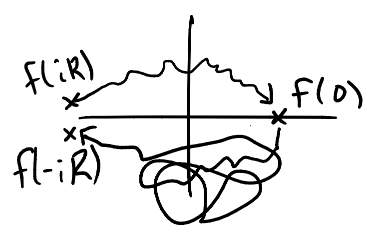
n odd: $f(iR), f(-iR)$

For $t > 0$, $\operatorname{Im}(f(it)) > 0$

for $t < 0$, $\operatorname{Im}(f(it)) < 0$

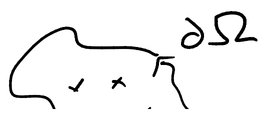
for $t = 0$, $f(0) = \beta^2$

total: $n \cdot 2\pi - 2\pi \Rightarrow$ exactly $n-1$ zeros



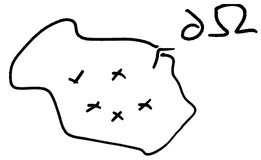
one negative rotation
net change in argument $\approx -2\pi$.

Rouché's theorem $\quad |f| > |g|$ on $\partial\Omega$



zeros − # poles of $f+g$

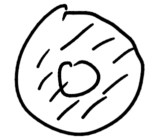
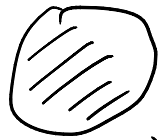
= # zero − # poles of f



zeros ...

= # zero − # poles of f

Ex: Count # zeros of $\underline{z^5+z^3+5z^2+2}$ in $1<|z|<2$

zeros in  = $\underbrace{\text{\# zeros in}}_{|z|=2}$  − $\underbrace{\text{\# zeros in}}_{|z|=1}$  3

$f \sim |z^5| = 32$

$g \sim |z^3+5z^2+2| \le 8+20+2 = 30$

zeros of $f+g$ in $|z|<2$

= # zeros of f in $|z|<2$

= 5

$f \to |5z^2| = 5$

$g \to |z^5+z^3+2| \le 4$

zeros of $f+g$

= # zeros of f

= 2

Residue Theorem

$$\underbrace{\int_\gamma f(z)\,dz}_{\int_a^b f(\gamma(t))\gamma'(t)\,dt} = 2\pi i \sum_{\substack{\text{singularities} \\ z_i \text{ inside } \gamma}} \mathrm{Res}[f(z), z=z_i]$$

$f(z) = \sum_{k\in\mathbb{Z}} a_k (z-z_i)^k$

residue $= a_{-1}$

$\int_0^\infty$ or $\int_{-\infty}^\infty$ $\frac{1}{\text{poly}}$ or $\frac{\sin}{\text{poly}}\,dx$

$\int_0^{2\pi}$ trig stuff

$$\int_0^\infty \frac{1}{1+x^5}\,dx$$

Use contour 

Use function $\frac{1}{1+z^5}$

$\int^R 1 \quad \int \frac{1}{\quad}\,dz + \int \frac{1}{\quad}\,dz = 2\pi i \sum$

$1+z^5$

$$\underbrace{\int_0^R \frac{1}{1+z^5}\,dz}_{\substack{z=t\\ 0\le t\le R}} + \underbrace{\int_{|z|=R} \frac{1}{1+z^5}\,dz}_{\substack{z=Re^{it}\\ 0\le t\le 2\pi/5}} + \underbrace{\int_{e^{2\pi i/5}R}^{0} \frac{1}{1+z^5}\,dz}_{\substack{z=e^{2\pi i/5}t\\ 0\le t\le R}} = 2\pi i \sum{}'$$

$$\int_0^R \frac{1}{1+t^5}\,dt + \int_0^{2\pi/5} \frac{1}{1+R^5e^{5it}} Rie^{it}\,dt - \int_0^R \frac{1}{1+t^5} e^{2\pi i/5}\,dt$$

$$=\left(1-e^{2\pi i/5}\right)\int_0^R \frac{1}{1+x^5}\,dx + \int_0^{2\pi/5} \frac{Rie^{it}}{1+R^5e^{5it}}\,dt = 2\pi i \sum{}' \mathrm{Res}$$

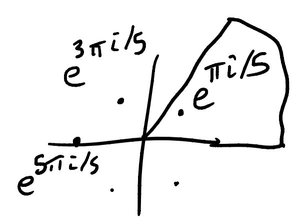
$$|Rie^{it}|=R, \qquad |R^5e^{5it}|=R^5$$

$$|1+R^5e^{5it}| \ge R^5-1$$

$$\int_0^{2\pi/5} \cdots\, dt \le \frac{2\pi}{5}\cdot\frac{R}{R^5-1} \to 0 \quad \text{as} \quad R\to\infty$$

$$\int_0^\infty \frac{1}{1+x^5}\,dx = \frac{2\pi i}{1-e^{2\pi i/5}}\sum{}' \mathrm{Res} = \frac{2\pi i}{1-e^{2\pi i/5}}\,\mathrm{Res}\left[\frac{1}{1+z^5},\ z=e^{\pi i/5}\right]$$

$\frac{1}{1+z^5}$ has singularities when $1+z^5=0$



Residue

$$\frac{1}{1+z^5} = \frac{1}{5e^{4\pi i/5}}\left(z-e^{\pi i/5}\right)^{-1} + \cdots$$

$$1+z^5 = a_1\left(z-e^{\pi i/5}\right) + a_2\left(z-e^{\pi i/5}\right)^2 + \cdots$$

$$a_1 = 5e^{4\pi i/5}$$

$\left(\frac{2\pi i}{5}\right) e^{-4\pi i/5}$

$a_1 = 5e$

$$\boxed{\frac{2\pi i}{1-e^{2\pi i/5}} \cdot \frac{1}{5e^{4\pi i/5}}} = \frac{\left(\frac{2\pi i}{5}\right) e^{-4\pi i/5}}{1-e^{2\pi i/5}}$$

mult by $e^{-\pi i/5}$

$$= \frac{\left(\frac{2\pi i}{5}\right)(-1)}{e^{-\pi i/5}-e^{\pi i/5}}$$

$$= \frac{\pi/5}{\frac{e^{\pi i/5}-e^{-\pi i/5}}{2i}} = \boxed{\frac{\pi/5}{\sin(\pi/5)}}$$

Ex: $0 < a < b$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|ae^{i\theta}-b|^4}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{(ae^{i\theta}-b)^2\,\overline{(ae^{i\theta}-b)}^2}\, d\theta$$

$$\int_{|z|=1} f(z)\,dz = \int_0^{2\pi} f(e^{it})\, ie^{it}\, dt$$

$z = e^{it}$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{(ae^{i\theta}-b)^2(ae^{-i\theta}-b)^2}\, d\theta$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{i\theta}\, ie^{i\theta}\, d\theta}{(ae^{i\theta}-b)^2(a-be^{i\theta})^2}$$

$$= \frac{1}{2\pi i}\int_{|z|=1} \frac{z\, dz}{(az-b)^2(a-bz)^2} = \operatorname{Res}\left[\frac{z}{(az-b)^2(a-bz)^2},\ z = \frac{a}{b}\right]$$

$$\frac{1}{(a-bz)^2} = \frac{1}{b^2}\left(z-\frac{a}{b}\right)^{-2}$$

$$\frac{z}{(az-b)^2} = \underline{\qquad}\left(z-\frac{a}{b}\right)^0 + \underline{\qquad}\left(z-\frac{a}{b}\right)^1 + \cdots$$

$$\frac{z}{(az-b)^2} = \underline{\quad\quad}\left(z-\frac{a}{b}\right)^0 + \underline{\quad\quad}\left(z-\frac{a}{b}\right)^1 + \cdots$$

$$\frac{d}{dz}\frac{z}{(az-b)^2} \quad \text{at} \quad z=\frac{a}{b}$$

$$\frac{(az-b)^2 - 2za(az-b)}{(az-b)^4} \quad \text{at } z=\frac{a}{b}$$

$$\boxed{\frac{1}{b^2}}\ \frac{\left(\frac{a^2}{b}-b\right)^2 - 2\frac{a^2}{b}\left(\frac{a^2}{b}-b\right)}{\left(\frac{a^2}{b}-b\right)^4}$$

$$= \frac{(a^2-b^2)^2 - 2a^2(a^2-b^2)}{(a^2-b^2)^4}$$

$$= \frac{(a^2-b^2)-2a^2}{(a^2-b^2)^3} = -\frac{a^2+b^2}{(a^2-b^2)^3}$$

$$= \frac{a^2+b^2}{(b^2-a^2)^3}$$

Schwarz Reflection

If $f(z) \in \mathbb{R}$ for $z \in \mathbb{R}$,

then $f(\bar{z}) = \overline{f(z)}$ (e.g., $\sin(\bar{z}) = \overline{\sin(z)}$
$e^{\bar{z}} = \overline{e^z}$)

(Why? $f(z)$ vs $\overline{f(\bar{z})}$

both holomorphic, agree on $\mathbb{R}$,

identity thm $\Rightarrow$ $f(z) = \overline{f(\bar{z})}$).