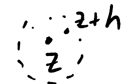
$U \subseteq \mathbb{C}$ open



Definitions $f : U \to \mathbb{C}$

- $f'(z) = \lim_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$ exists

at every point of U

(Apologies for the mess!)

- $f(z) = u(x+iy) + iv(x+iy)$

$u(x,y)$ $v(x,y)$

Cauchy-Riemann equations

$u_x = v_y \quad u_y = -v_x$

- Holomorphic: Differentiable and derivative is continuous
- Infinitely differentiable
- Analytic: $f(z)$ is locally a power series

at every $z_0 \in U$, there is a

st

$$f(z) = \sum_{k \geq 0} (z - z_0)^k \text{ on nbhd.}$$

- Morera's theorem

$$\int_\Gamma f(z)\,dz = 0$$

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for all small triangles

$\gamma : [a,b] \to \mathbb{C}$

$$\int_\Gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

Only need to check

Γ is a square (or triangles)

f_n sequence of analytic functions

$f_n : U \to \mathbb{C}$ $f_n \to f$ uniform convergence

$\forall \varepsilon, \exists n_0$ $|f_n - f| < \varepsilon$ for all $n \geq n_0$.

Show f is analytic

We know that for each $\Gamma \in U$,

$$\int_\Gamma f_n(z) = 0$$

$$\left| \int_\Gamma f_n(z)\,dz - \int_\Gamma f(z)\,dz \right|$$

$$= \left| \int_\Gamma f_n(z) - f(z)\,dz \right|$$

$$= \left| \int_a^b f_n(\gamma(t)) - f(\gamma(t))\,dt \right| \quad \gamma'(t)$$

$$\leq \int_a^b |f_n(\gamma(t)) - f(\gamma(t))|\,dt$$

$$\leq \int_a^b \varepsilon\,|\gamma'(t)|\,dt$$

$$= \varepsilon \cdot \text{length}(\Gamma)$$

If $\int_\Gamma f_n = 0$

$\Rightarrow \int_\Gamma f = 0$

$\Rightarrow f$ is analytic

Open mapping theorem:

If $f : U \to \mathbb{C}$ is holomorphic

then $f(U)$ is open

then $f(U)$ is open
(unless f is constant)

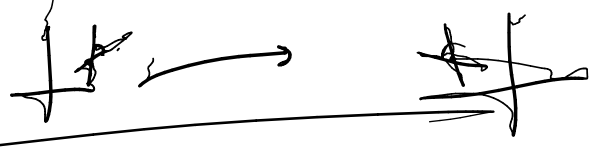
Conformal Maps $f: U \to V \subseteq \mathbb{C}$

f is bijective

$f'(z) \neq 0$

f is biholomorphic ($f^{-1}: V \to U$ holomorphic)

f is angle preserving



Riemann mapping theorem

If $U \subseteq \mathbb{C}$ is nonempty,
$U \neq \mathbb{C}$, simply connected (no holes)
then U is conformally equivalent to $\mathbb{D}$.

Möbius Transformations / Linear Fractional Transformations

$\frac{az+b}{cz+d} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ $\quad \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$a,b,c,d \in \mathbb{C}$ $\quad \infty \mapsto \frac{a}{c}$ $\quad$ - determined by 3 points

$ad-bc \neq 0$ $\quad -\frac{d}{c} \mapsto \infty$ $\quad$ - send circles/lines to circles/lines

$$\left(\frac{az+b}{cz+d}\right) \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

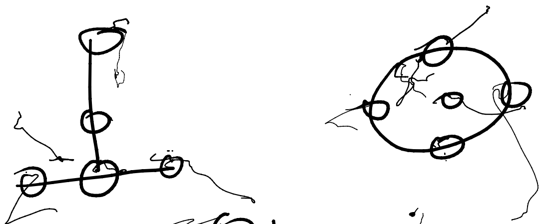
$$\left(\frac{az+b}{cz+d}\right) \circ \left(\frac{ez+f}{gz+h}\right) \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$(cz+d)\,(gz+h)$

$$\left(\frac{az+b}{cz+d}\right)^{-1} \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$

$f: \mathbb{H} \longrightarrow \mathbb{D}$

$$\frac{z-i}{z+i} \qquad \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}^{-1}$$



$0 \longmapsto -i$

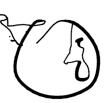
$\infty \longmapsto i$

$i \longmapsto 0$

$1 \longmapsto 1$

$-1 \longmapsto -1$

$$i\,\frac{z-i}{z+i}$$



$$i \quad \frac{z-i}{z+i}$$

$\mathbb{D} \longrightarrow \mathbb{H} \longrightarrow \mathbb{D}$

Schwarz lemma

If $f: \mathbb{D} \to \mathbb{D}$, $f(0)=0$

then $|f(z)| \le |z|$.

Conformal $\mathbb{D} \longrightarrow \mathbb{D}$ is $e^{i\theta}\,\dfrac{z-a}{1-\bar{a}z}$

$a \longmapsto 0$

Conformal $\mathbb{C} \longrightarrow \mathbb{C}$ is $az+b$ with $a \neq 0$

Conformal $\hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ is $\dfrac{az+b}{cz+d}$ with $ad-bc \neq 0$

Conformal $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is $\frac{az+b}{cz+d}$ with $ad-bc \neq 0$

(any $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational function $\frac{\text{poly}}{\text{poly}}$)

If $f: B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is analytic

$\to \mathbb{C}$

then $f(z) = \sum_{k \in \mathbb{Z}} a_k (z-z_0)^k$ for $a_k \in \mathbb{C}$

(Laurent Series)

- If $a_k = 0$ for all $k<0$
  - Removable singularity
  - Can extend $f: B_r(z_0) \to \mathbb{C}$
  - Iff f is bounded near z_0 (Riemann's theorem)
- If $a_k = 0$ for all $k < k_0 < 0$
  - Pole
  - Iff $f(z) \to \infty$ as $z \to z_0$

Same from the $\hat{\mathbb{C}}$ perspective

- If $a_k \neq 0$ for infinitely many $k<0$
  - Essential Singularity
  - Casorati-Weierstrass: Image of $B_\varepsilon(z_0)$ is dense in $\mathbb{C}$

Liouville's Theorem: A bounded entire function is constant.