Basics

A metric $(d(x,x)=0,\ d(x,y)=d(y,x),\ d(x,z)\le d(x,y)+d(y,z))$ on a set X induces a topology w/ base

$$B_r(x)=\{y\in X \mid d(x,y)<r\}$$

i.e. $\mathcal{U}\subseteq X$ is <u>open</u> if for all $x\in\mathcal{U}$ there is $r_x>0$ s.t. $B_{r_x}(x)\subseteq\mathcal{U}$. The complement of an open set is <u>closed</u>.

- Arbitrary union of opens are open
- finite intersections of opens are open.

Notion of convergence generalizes directly to metric spaces. A set $C\subseteq X$ is closed $\iff$ if $\{x_n\}\subseteq C$ converges to $x\in X$, then $x\in C$.

Note metric spaces are <u>Hausdorff</u>, that is, for any distinct $x,y\in X$ there are open sets $x\in\mathcal{U}_x$, $y\in\mathcal{U}_y$ s.t.

$$\mathcal{U}_x\cap\mathcal{U}_y=\emptyset$$

e.g. take open balls. This implies that sequences can only converge to a single element.

For X, Y metric spaces, the following definitions of a continuous map $f: X \to Y$ are equivalent

i) For each $x \in X$, for any $\varepsilon > 0$ there is $\delta > 0$ s.t.

$$d(y,x) < \delta \implies d(f(y), f(x)) < \varepsilon$$

ii) For any $U \subseteq Y$ open, $f^{-1}(U)$ is open

iii) For any convergent sequence $\{x_n\} \subseteq X$ $x_n \to x$, then $f(x_n) \to f(x)$

4.1.13 - Fall 1986 6

Let $\{y_n\} \subseteq Y$ converge to $y \in X$. Suppose $d(x,y) < r$.

Then $r = d(x, y_n) \le d(x,y) + d(y, y_n) \implies d(y, y_n) \ge r - d(x,y) > 0$

Then $B_{r-d(x,y)}(y)$ contains none of the y_n, a contradiction, so $y \in Y$.

✓

In metric spaces there is a sequential formulation of compactness.

<u>Thm</u> For $Z \subseteq X$ the following are equivalent

i) Any collection of open sets $\{\mathcal{U}_\alpha\}$ s.t. $Z \subseteq \bigcup_\alpha \mathcal{U}_\alpha$, there is a finite subset $\{\mathcal{U}_i\}$ s.t. $Z \subseteq \bigcup_{i=1}^{n} \mathcal{U}_i$

ii) Every sequence in Z has a convergent subsequence in Z.

Compact $\Rightarrow$ closed and bounded. In $\mathbb{R}^n$ the converse is true.

Compactness is topological so its preserved under continuous maps.

<u>Thm</u> If $f : X \longrightarrow Y$ is continuous, $Z \subseteq X$ is compact, then $f(Z)$ is compact.

pf/ Take preimage of open cover of $f(Z)$ and take finite subcover $\square$

4.2.6 - Fall 1980 19

Suppose for contradiction there is $x \notin f(X)$. X is compact so $f(X)$ also is, and in particular closed. Thus there is $\varepsilon > 0$ s.t. $d(x, f(X)) \geq \varepsilon$.

Now consider the sequence $\{f^n(x)\}$ which must have a convergent subsequence. However, for any $m > n$

$$d(f^n(x), f^m(x)) = d(x, f^{m-n}(x)) \geq \varepsilon$$

so there can't be a convergent subsequence ✓

There is a stronger version of continuity only available in metric spaces:

<u>Def</u> $f : X \to Y$ is uniformly continuous if for any $\varepsilon > 0$ there is $\delta > 0$ s.t.

$$d(x,y) < \delta \implies d(f(x), f(y)) < \varepsilon$$

<u>Thm</u> If $f : X \to Y$ is continuous and X is compact, then f is uniformly continuous.

pf/ $\varepsilon>0$. For each $x \in X$ consider $\delta_x > 0$ s.t.
$f(B_{\delta_x}(x)) \subseteq B_\varepsilon(f(x))$. Then take subcover $B_{\delta_i}(x_i)$
and set $\delta = \min(\delta_i)$.

$\square$

4.1.18 - Fall 1989 14

$f(X)$ is compact and thus bounded, so
$$|f(x)-f(y)| < B \quad \text{for all } x,y.$$
As long as $|x-y| \geq c$ and $M \geq B/c$, then for any $\varepsilon>0$
$$|f(x)-f(y)| \leq \varepsilon + M|x-y|$$
f is uniformly continuous so for $\varepsilon>0$ there is $\delta>0$ s.t.
$$|f(x)-f(y)| < \varepsilon \quad \text{for } |x-y|<\delta$$
Thus set $M = B/\delta$ works ✓

Space of Continuous Functions

Now consider $C(X) := \{f : X \to \mathbb{R} \text{ continuous}\}$ where X is a compact metric space.

This is a normed vector space with

$$\|f\| = \max |f|$$

$d(f,g) := \|f-g\|$ defines a metric on $C(X)$.

If $f_n \to f$ in $C(X)$ that means

$$\max |f - f_n| \to 0$$

that is, f_n converges uniformly to f.

Recall the result of a uniformly converging sequence of continuous functions is always continuous. Thus, since $\mathbb{R}^n$ is a complete metric space (all cauchy sequences converge), so is $C(X)$.

There is a characterization of compact sets in $C(X)$.

Defn A subset $\mathcal{F} \subseteq C(X)$ is <u>equicontinuous</u> if for each $x \in X$, for every $\varepsilon > 0$ there is $\delta > 0$ s.t.

$$|f(x') - f(x)| < \varepsilon$$

for all $d(x', x) < \delta$, for all $f \in \mathcal{F}$.

Thm (Arzela-Ascoli) $\mathcal{F} \subseteq C(X)$ is s.t. $\overline{\mathcal{F}}$ is compact $\iff$ $\mathcal{F}$ is uniformly bounded and equicontinuous.

<u>4.2.10 - Spring 1987 16</u>

Let $\varepsilon > 0$, $x \in X$. There is $f_1 \in \mathcal{F}$ s.t.

$$f_1(x) \leq g(x) \leq f_1(x) + \varepsilon$$

Let $\delta > 0$ s.t. if $y \in B_\delta(x)$ then

$$|f(x) - f(y)| < \varepsilon \quad \forall f \in \mathcal{F}$$

Then for $y \in B_\delta(x)$ there is $f_2 \in \mathcal{F}$ s.t.

$$f_2(y) \leq g(y) \leq f_2(y) + \varepsilon$$

By definition of g

$$f_2(x) \leq f_1(x) + \varepsilon$$

by equicontinuity f_2 can increase by at most ε, f_1 can decrease by at most ε.

$$f_2(y) \le f_1(y) + 3\varepsilon$$

thus

$$f_1(y) \le g(y) \le f_1(y) + 3\varepsilon$$

$\Rightarrow$

$$|g(y) - g(x)| \le 2\varepsilon$$

for all $y \in B_\delta(x)$ ✓

<u>Thm</u> (Banach fixed-point) Suppose X is a complete metric space and $T: X \to X$ is a map s.t. there is $q \in [0,1)$ with

$$d(T(x), T(y)) \le q\, d(x,y)$$

for all $x, y \in X$. Then there is a <u>unique</u> $x^* \in X$ s.t. $T(x^*) = x^*$

<u>Fall 1982 18</u> Consider the map

$$T: C([0,1]) \longrightarrow C([0,1])$$

$$f \longmapsto e^{x^2} - \int_0^1 K(x,y) f(y)\,dy$$

Looking for a fixed point. We have

$$\|Tf - Tg\| = \max \left| \int_0^1 K(x,y)\big(g(y) - f(y)\big)dy \right|$$
$$\leq \max |K(x,y)|\; \|f - g\|$$

by assumption $\max |K(x,y)| < 1$ so done by fixed point theorem.