Metric Spaces

Outline

Basics

- Definition, open/closed sets, convergence, Hausdorff
- Compactness, continuous maps

More about continuous functions

- Uniform continuity
- The space $C(X)$, X a compact metric space
- Equicontinuity and Arzela-Ascoli theorem

Misc.

- Banach fixed point theorem/contraction mapping principle
- Diagonalization arguments

Problems

4.1.13 - Fall 1989 6 Let $X \subset \mathbb{R}^n$ be a closed set and r a fixed positive real number. Let $Y = \{y \in \mathbb{R}^n | |x - y| = r, \text{ some } x \in X\}$. Show that Y is closed.

4.2.6 - Fall 1980 19 Let X be a compact metric space and $f : X \to X$ an isometry. Show that $f(X) = X$.

4.1.18 - Fall 1989 14 Let $X \subset \mathbb{R}^n$ be compact and let $f : X \to \mathbb{R}$ be continuous. Given $\epsilon > 0$, show there is an M such that for all $x, y \in X$,

$$|f(x) - f(y)| \leq M|x - y| + \epsilon.$$

4.2.10 - Spring 1987 16 Let $\mathcal{F}$ be a uniformly bounded, equicontinuous family of real valued functions on the metric space (X, d). Prove that the function

$$g(x) = \sup\{f(x) | f \in \mathcal{F}\}$$

is continuous.

4.3.5 - Fall 1982 18 Let K be a continuous function on the unit square $0 \leq x, y \leq 1$ satisfying $|K(x, y)| < 1$ for all x and y. Show that there is a continuous function $f(x)$ on $[0, 1]$ such that we have

$$f(x) + \int_0^1 K(x, y) f(y) dy = e^{x^2}$$

can there be more than one such function f?