Similarity V a finite dimensional vector space

A basis $v_1, \dots, v_n$ for V induces an isomorphism

$$V \longrightarrow F^n$$
$$v_i \longmapsto e_i$$

So, if $T: V \to V$ a linear map, a basis $\mathcal{B}$ for V induces a linear map $T_{\mathcal{B}}: F^n \to F^n$

If $\mathcal{C}$ is a different basis, how does $T_{\mathcal{C}}$ compare to $T_{\mathcal{B}}$? A: There exists an isomorphism $P: F^n \to F^n$ s.t.

$$T_{\mathcal{C}} = P^{-1} T_{\mathcal{B}} P$$

Defn Two square matrices A and B are called similar if there is an invertible matrix P s.t.

$$B = P^{-1} A P$$

Diagonalizable means similar to a diagonal matrix.

Rational Canonical Form Let R be a PID.

There is a classification of finitely generated R-modules M:

<u>Thm</u> 1) $M \cong R^r \oplus R/(a_1) \oplus \cdots \oplus R/(a_m)$

$r \geq 0$, $a_1, \ldots, a_m \in R$ non units s.t. $a_1 | \ldots | a_m$

2) $r, a_1, \ldots, a_m$ unique up to units.

Now, R is a UFD, so can write $a \in R$ as

$$a = u p_1^{\alpha_1} \cdots p_s^{\alpha_s} \quad \begin{array}{l} u \text{ a unit} \\ p_1, \ldots, p_s \text{ prime} \end{array}$$

Then by Chinese remainder theorem

$$R/(a) \cong R/(p_1^{\alpha_1}) \oplus \cdots \oplus R/(p_s^{\alpha_s})$$

<u>Thm</u> If M a fin gen R-module

$$M \cong R^r \oplus R/(p_1^{\alpha_1}) \oplus \cdots \oplus R/(p_s^{\alpha_s})$$

$r \geq 0$, $p_1^{\alpha_1}, \ldots, p_s^{\alpha_s}$ powers of possibly nondistinct primes, also unique up to units.

Now, if $T: V \to V$ a linear map, we can turn V into an $F[x]$-module by having x act by T i.e.

$$(a_n x^n + \dots + a_1 x + a_0) v$$
$$:= a_n T^n v + \dots + a_1 T v + a_0 v$$

V is finite dimensional so this is a finitely generated $F[x]$-module so

$$V = F[x] / (a_1(x)) \oplus F[x] / (a_2(x)) \dots \oplus F[x] / (a_m(x))$$

$$a_1(x) | \dots | a_m(x)$$ unique if require the a_i to be monic

Note: a_m is the minimal polynomial.

If $a(x) = x^k + b_{k-1} x^{k-1} + \dots + b_1 x + b_0$, then $1, \bar{x}, \dots, \bar{x}^{k-1}$ is a basis for the F-vector space $F[x] / (a(x))$.

On this basis multiplication acts by

$$1 \mapsto \bar{x}$$
$$\bar{x} \mapsto \bar{x}^2$$
$$\vdots$$

$$\overline{x}^{k-2} \longmapsto \overline{x}^{k-1}$$

$$\overline{x}^{k-1} \longmapsto -b_0 - b_1\overline{x} - \cdots - b_{k-1}\overline{x}^{k-1}$$

which has matrix representation

$$C_a := \begin{pmatrix} 0 & 0 & \cdots & & -b_0 \\ 1 & 0 & \ddots & & -b_1 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -b_{k-1} \end{pmatrix}$$

Thus there is a basis of V s.t. T becomes

$$\begin{pmatrix} C_{a_1} & & & \\ & C_{a_2} & & \\ & & \ddots & \\ & & & C_{a_m} \end{pmatrix}$$

This is the rational canonical form of T.

This is the canonical form of T. Two matrices over F are similar $\Leftrightarrow$ they have the same rational canonical form,

Note the characteristic polynomial of C_a is $\pm a$, thus the characteristic polynomial of T is $\pm a_1 \cdots a_m$ which proves

Cayley Hamilton and that the characteristic poly and minimal polynomial have the same factors.

E.g. $A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}$ Char poly is $(x-2)^2(x-3)$
minimal poly is $(x-2)(x-3)$

Thus $a_1 = (x-2)$, $a_2 = (x-2)(x-3) = x^2 - 5x + 6$

so rational canonical form is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & 1 & 5 \end{pmatrix}$$

<u>7.2.10</u> Let R_A, R_B the rational canonical forms of A and B respectively. Since A and B are real, the rational canonical forms over $\mathbb{C}$ are the same as over $\mathbb{R}$. Then $R_A = R_B$ since we assumed A, B similar over $\mathbb{C}$, thus they are also similar over $\mathbb{R}$. ✓

<u>Jordan Canonical Form</u>

Let $T: V \longrightarrow V$ linear. Assume minimal polynomial factors into linear terms (always true over $\mathbb{C}$). Then factoring $a_1, \dots, a_m$ gives

$$V \cong F[x]/(x-\lambda_1)^{k_1} \oplus \dots \oplus F[x]/(x-\lambda_\ell)^{k_\ell}$$

λ_i may be repeated but they are each roots of the minimal polynomial (i.e. eigenvalues)

with respect to the basis $(\bar{x}-\lambda)^{k-1}, \dots, \bar{x}-\lambda, 1$

x acts on $F[x]/(x-\lambda)^k$ as

$$k \times k \quad \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$$

This is a Jordan block of size k with eigenvalue λ

Such a Jordan block has minimal polynomial $(x-\lambda)^k$.
It has only λ as an eigenvalue w/ dim of eigenspace 1.

T can be represented as a matrix

$$\begin{pmatrix} J_1 & & & \\ & \ddots & & O \\ & & \ddots & \\ O & & & J_\ell \end{pmatrix}$$

where J_i is a Jordan block of size k_i w/ eigenvalue λ_i. This is <u>Jordan canonical form</u> unique up to permutation. Note the minimal polynomial of T is the least common multiple of the $(x-\lambda_i)^{k_i}$

<u>7.6.24</u> The char poly of A is $-(x-1)^3$. The eigenvalues are just 1. The eigenvectors are x s.t.

$$(A - I)x = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} x = 0$$

These are vectors of the form

$$\begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\text{basis for eigenspace}}$$

Minimal poly divides char poly so it's $(x-1), (x-1)^2$ or $(x-1)^3$.

$$(A-I)^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^2 = 0$$

so minimal poly is $(x-1)^2$. There is then a Jordan block of size 2 so

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the Jordan canonical form.

__7.6.30__ Triangular so char poly is $(x-1)^6$. Minimal poly is then a power of $(x-1)$. Direct calculation gives

$$(A-I)^2 \neq 0, \quad (A-I)^3 = 0$$

so minimal poly is $(x-1)^3$. Thus there is a 3×3 block.

To determine the rest, find eigenspace of 1: Solve $Ax = x$

i.e.

$$x_1 = x_1$$
$$x_1 + x_i = x_i \quad 2 \le i \le 5$$
$$x_1 + x_2 + \dots + x_6 = x_6$$

$\Rightarrow x_1 = 0$, $x_2 + x_3 + x_4 + x_5 = 0$, x_6 undetermined

so 4 degrees of freedom. Thus 3 1×1 blocks and 1 3×3 block, all eigenvalue 1

✓

7.2.6 By assumption they both satisfy $x(x-1)$.

Possible minimal poly's are

$$x \quad , \quad x-1 \quad , \quad x(x-1)$$

0 matrix   Identity $0 < \text{rank} < n$

Since minimal poly is least common multiple of minimal polys of Jordan blocks, each Jordan block is 1×1.

Then if they have the same rank they have the same Jordan canonical forms, so they're similar.

✓

The Jordan Canonical form gives

<u>Thm</u> $A \in M_{n\times n}(F)$ is diagonalizable $\Leftrightarrow$ m_A factors into linear terms in F w/ no repeated roots

<u>Thm</u> $A \in M_{n\times n}(F)$ triangularizable $\Leftrightarrow$ m_A (equivalently p_A) factors into linear terms over F.

All complex matrices are triangularizable and has an eigenvector.

<u>7.5.7</u> $M_{n\times n}(\mathbb{C})$ is fin dim and matrices commuting with a fixed matrix $A \in M_{n\times n}(\mathbb{C})$ is a subspace. Thus can assume S is finite. Go by induction. Suppose $A_1, \ldots, A_n$ commuting and $A_1, \ldots, A_{n-1}$ have a common eigenvector. Let E be the space of all such common eigenvectors.

Let $v \in E$. Then $A_i A_{n+1} v = A_{n+1} A_i v = \lambda_i A_{n+1} v$ for all i so $A_{n+1} v \in E$. Then can view A_{n+1} as a map

$$E \to E$$

since A_{n+1} is complex, it has an eigenvector in E.

✓

<u>7.6.17</u> Changing basis we can write T as a matrix

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

where A is the matrix of $T|_W : W \to W$

Want to show m_A factors into linear terms w/ none repeated. The matrix of T^k is

some thing

$$\begin{pmatrix} A^k & - \\ 0 & B^k \end{pmatrix}$$

so any polynomial satisfied by T is also satisfied by A. Thus $m_A \mid m_T$ and must also factor into linear terms since m_T does. ✓

Inner/Hermitian Products

Last time we covered inner products on real vector spaces, the spectral theorem was if $T: V \to V$ a linear endomorphism of a real inner product space V, then

$$T^* = T$$

$$\iff$$

T has an orthonormal basis of eigenvectors

$$\iff$$

There is a matrix U s.t. $U^T = U^{-1}$ and

$$\text{diagonal} \longrightarrow D = U^T A U$$

where A is a representation of T in an orthonormal basis.

If V is a complex vector space then there is no bilinear (over $\mathbb{C}$), symmetric, positive definite form on V. Instead define Hermitian inner product

$$\langle\,,\,\rangle : V \times V \to \mathbb{C}$$

to be s.t.

i) $\langle av_1 + bv_2, w\rangle = a\langle v_1, w\rangle + b\langle v_2, w\rangle$

$\langle v, aw_1 + bw_2\rangle = \bar{a}\langle v, w_1\rangle + \bar{b}\langle v, w_2\rangle$

ii) $\langle v, w\rangle = \overline{\langle w, v\rangle}$

iii) $\langle v, v\rangle \geq 0$ equality $\iff v = 0$

$\mathbb{C}^n$ has a standard hermitian inner product

$$\langle z, w\rangle = z_1\bar{w}_1 + \dots + z_n\bar{w}_n$$

Then if $T : \mathbb{C}^n \to \mathbb{C}^m$ linear, its <u>Hermitian adjoint</u> is $T^* : \mathbb{C}^m \to \mathbb{C}^n$ defined by

$$\langle Tv, w\rangle = \langle v, T^* w\rangle$$

if A is the matrix of T, then the matrix of T^* is $\bar{A}^T =: A^*$

<u>Thm</u> (Spectral thm over $\mathbb{C}$) $T: \mathbb{C}^n \to \mathbb{C}^m$ is <u>normal</u> i.e. $TT^* = T^*T \iff T$ has an orthonormal basis of eigenvectors.

Note that <u>Hermitian</u> maps (i.e. T s.t. $T = T^*$ complex version of symmetric) are normal.

Gram-Schmidt: From a basis $v_1, \ldots, v_n$ we can obtain an orthonormal basis $e_1, \ldots, e_n$ by the procedure

$$e_1 = \frac{v_1}{\|v_1\|}$$

$$e_2 = \frac{v_2 - \langle v_2, e_1 \rangle}{\|v_2 - \langle v_2, e_1 \rangle\|}$$

$$\vdots$$

$$e_n = \frac{v_n - \langle v_n, e_1 \rangle - \ldots - \langle v_n, e_{n-1} \rangle}{\| v_n - \langle v_n, e_1 \rangle - \ldots - \langle v_n, e_{n-1} \rangle \|}$$