Similarity: V a finite dimensional vector space. A basis $v_1, ..., v_n$ for $V$ induces an isomorphism

$$V \rightarrow F^n$$

$$v_i \rightarrow e_i$$

So, if $T : V \rightarrow V$ a linear map, a basis $B$ for $V$ induces a linear map $T_B : F^n \rightarrow F^n$.

If $Z$ is a different basis, how does $T_Z$ compare to $T_B$? $A$: There exists an isomorphism $P : F^n \rightarrow F^n$ s.t.

$$T_Z = P^{-1} T_B P$$

**Defn** Two square matrices $A$ and $B$ are called similar if there is an invertible matrix $P$ s.t.

$$B = P^{-1} A P$$

Diagonalizable means similar to a diagonal matrix.

**Rational Canonical Form** Let $R$ be a PID.
There is a classification of finitely generated $R$-modules $M$:

**Thm** 1) $M \cong R^r \oplus R/(a_1) \oplus \ldots \oplus R/(a_m)$

$r \geq 0, a_1, \ldots, a_m \in R$ non units s.t. $a_1 \mid \ldots \mid a_m$

2) $r, a_1, \ldots, a_m$ unique up to units.

Now, $R$ is a UFD, so can write $a \in R$ as

$a = u p_1^{r_1} \ldots p_s^{r_s}$

where $u$ is a unit, $p_1, \ldots, p_s$ prime.

Then by Chinese remainder theorem

$R/(a) \cong R/(p_1^{r_1}) \oplus \ldots \oplus R/(p_s^{r_s})$

**Thm** If $M$ a fin gen $R$-module

$M \cong R^r \oplus R/(p_1^{r_1}) \oplus \ldots \oplus R/(p_s^{r_s})$

$r \geq 0, p_1^{r_1}, \ldots, p_s^{r_s}$ powers of possibly nondistinct primes, also unique up to units.
Now, if $T: V \to V$ a linear map, we can turn $V$ into an $F[x]$-module by having $x$ act by $T$, i.e.,

$$(anx^n + \ldots + a_1x + a_0)v = a_nTv + \ldots + a_1Tv + a_0v$$

$V$ is finite dimensional so this is a finitely generated $F[x]$-module so

$$V = \frac{F[x]}{(a_1(x))} \oplus \frac{F[x]}{(a_2(x))} \oplus \cdots \oplus \frac{F[x]}{(a_m(x))}$$

$a_1(x), \ldots, a_m(x)$ unique if require the $a_i$ to be monic.

Note: $a_m$ is the minimal polynomial.

If $a(x) = x^k + b_{k-1}x^{k-1} + \ldots + b_1x + b_0$, then $1, x, \ldots, x^{k-1}$ is a basis for the $F$-vector space $\frac{F[x]}{(a(x))}$.

On this basis multiplication acts by

$1 \mapsto x$,

$x \mapsto x^2$,

\vdots
\[
x^{k-2} \mapsto x^{k-1}
\]
\[
x^{k-1} \mapsto -b_0 - b_1 x - \ldots - b_{k-1} x^{k-1}
\]

which has matrix representation

\[
C_a := \begin{pmatrix}
0 & 0 & \cdots & -b_0 \\
0 & 0 & \cdots & -b_1 \\
0 & 0 & \cdots & \ddots \\
0 & 0 & \cdots & 1 - b_{k-1}
\end{pmatrix}
\]

Thus there is a basis of \( V \) s.t. \( T \) becomes

\[
\begin{pmatrix}
C_a_1 \\
C_a_2 \\
\vdots \\
C_a_m
\end{pmatrix}
\]

This is the rational canonical form of \( T \).

This is the canonical form of \( T \). Two matrices over \( F \) are similar \( \iff \) they have the same rational canonical form.

Note the characteristic polynomial of \( C_a \) is \( \pm a \), thus the characteristic polynomial of \( T \) is \( \pm a_1 \ldots a_m \) which proves
Cayley Hamilton and that the characteristic poly and minimal polynomial have the same factors.

Eg. \( A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix} \) Char poly is \((x-2)^2(x-3)\)

minimal poly is \((x-2)(x-3)\)

Thus \( a_1 = (x-2) \), \( a_2 = (x-2)(x-3) = x^2 - 5x + 6 \)

so rational canonical form is

\[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 0 & -6 \\
0 & 1 & 5
\end{pmatrix}
\]

7.2.10 Let \( R_A, R_B \) the rational canonical forms of \( A \) and \( B \) respectively. Since \( A \) and \( B \) are real, the rational canonical forms over \( C \) are the same as over \( \mathbb{R} \). Then \( R_A = R_B \) since we assumed \( A, B \) similar over \( C \), thus they are also similar over \( \mathbb{R} \).

\( \checkmark \)

Jordan Canonical Form

Let \( T: V \rightarrow V \) linear. Assume minimal polynomial factors into linear terms (always true over \( C \)). Then
Factoring a... gives

\[ V = \frac{F[x]}{(x-\alpha)} \oplus \cdots \oplus \frac{F[x]}{(x-\beta)^k} \]

\( \alpha \) may be repeated but they are each roots of the minimal polynomial (i.e., eigenvalues) with respect to the basis \((x-\alpha)^{k-1}, \ldots, x-\alpha, 1\)

\( x \) acts on \( \frac{F[x]}{(x-\beta)^k} \) as

\[
\begin{pmatrix}
2 & 1 \\
2 & 1 \\
& & \ddots \\
& & & \ddots \\
& & & 2 & 1
\end{pmatrix}
\]

This is a Jordan block of size \( k \) with eigenvalue \( \beta \)

Such a Jordan block has minimal polynomial \((x-\beta)^k\).

It has only \( \beta \) as an eigenvalue w/ \( \dim \) of eigenspace 1.
T can be represented as a matrix

\[
\begin{pmatrix}
\bar{\delta}_i & \cdot & 0 \\
\cdot & \cdot & \cdot \\
0 & \cdot & \bar{\delta}_i
\end{pmatrix}
\]

where \( \bar{\delta}_i \) is a Jordan block of size \( k_i \) with eigenvalue \( \lambda_i \). This is the Jordan canonical form unique up to permutation. Note the minimal polynomial of \( T \) is the least common multiple of the \( (x-\lambda_i)^{k_i} \).

7.6.24 The char poly of \( A \) is \(-\lambda^3 \). The eigenvalues are just 1. The eigenvectors are \( x \) s.t.

\[
(A - I) x = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 
\end{pmatrix} x = 0
\]

These are vectors of the form

\[
\begin{pmatrix}
-x_1 - x_2 \\
x_2 \\
x_3
\end{pmatrix} = x_2 \begin{pmatrix}
-1 \\
1 \\
0
\end{pmatrix} + x_3 \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

basis for eigenspace
Minimal poly divides char poly so it's \((x-1), (x-1)^2\) or \((x-1)^3\).

\[
(A - I)^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^2 = 0
\]

so minimal poly is \((x-1)^2\). There is then a Jordan block of size 2 so

\[
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

is the Jordan canonical form.

7.6.30 Triangular so char poly is \((x-1)^4\).

Minimal poly is then a power of \((x-1)\). Direct calculation gives

\[
(A - I)^2 \neq 0, \quad (A - I)^3 = 0
\]

so minimal poly is \((x-1)^3\). Thus there is a 3x3 block.
To determine the rest, find eigenspace of $1$: Solve $Ax = x$

i.e.

\[
\begin{align*}
    x_1 &= x_1 \\
    x_1 + x_2 &= x_2 \\
    x_1 + x_2 + x_3 &= x_3 \\
    x_1 + x_2 + x_3 + x_4 &= x_4 \\
    x_1 + x_2 + x_3 + x_4 + x_5 &= x_5 \\
\end{align*}
\]

$\Rightarrow x_1 = 0$, $x_2 + x_3 + x_4 + x_5 = 0$, $x_6$ undetermined

so 4 degrees of freedom. Thus 3 $1 \times 1$ blocks

and 1 $3 \times 3$ block, all eigenvalue 1

**7.7.6** By assumption they both satisfy $x(x-1)$.

Possible minimal poly's are

\[
\begin{align*}
    x, & \quad x-1, & \quad x(x-1) \\
\end{align*}
\]

\[
\begin{align*}
    \uparrow & \quad \uparrow \\
    \text{O matrix} & \quad \text{Identity} \\
\end{align*}
\]

Since minimal poly is least common multiple of minimal poly of Jordan blocks, each Jordan block is $1 \times 1$.

Then if they have the same rank, they have the same Jordan canonical forms, so they're similar.
The Jordan Canonical Form gives

**Thm** \( A \in M_{n \times n}(F) \) is diagonalizable \( \iff \) \( m_A \) factors into linear terms in \( F \) with no repeated roots

**Thm** \( A \in M_{n \times n}(F) \) triangularizable \( \iff \) \( m_A \) (equivalently \( p_A \)) factors into linear terms over \( F \).

All complex matrices are triangularizable and have an eigenvector.

7.5.7 \( M_{n \times n}(C) \) is fin dim and matrices commuting with a fixed matrix \( A \in M_{n \times n}(C) \) is a subspace. Thus can assume \( S \) is finite.

Go by induction. Suppose \( A_1, \ldots, A_n \) commuting and \( A_1, \ldots, A_{n-1} \) have a common eigenvector.

Let \( E \) be the space of all such common eigenvectors.

Let \( v \in E \). Then \( A_i A_{n+1} v = A_{n+1} A_i v = 2 \xi A_{n+1} v \) for all \( i \) so \( A_i v \in E \). Then can view \( A_{n+1} \) as
a map \[ E \to E \]
since \( A \) is complex, it has an eigenvector in \( E \).

\[ \checkmark \]

7.6.17 Changing basis we can write
\( T \) as a matrix
\[
\begin{pmatrix}
A & C \\
0 & B
\end{pmatrix}
\]
where \( A \) is the matrix of \( T|_W : W \to W \)
Want to show \( m_A \) factors into linear terms w/ none repeated. The matrix of \( T^k \) is
\[
\begin{pmatrix}
A^k & \text{something} \\
0 & B^k
\end{pmatrix}
\]
so any polynomial satisfied by \( T \) is also satisfied by \( A \). Thus \( m_A | m_T \) and must
also factor into linear terms since \( M_T \) does.

**Inner/Hermition Products**

Last time we covered inner products on real vector spaces, the spectral theorem was:

\[
T: V \to V \text{ a linear endomorphism of a real inner product space } V, \text{ then } \\
T^* = T
\]

\( \iff \)

\( T \) has an orthonormal basis of eigenvectors

\( \iff \)

There is a matrix \( U \) s.t. \( U^T = U^{-1} \) and diagonal

\[ D = U^T A U \]

Where \( A \) is a representation of \( T \) in an orthonormal basis.

If \( V \) is a complex vector space then there is no bilinear (over \( \mathbb{C} \)), symmetric, positive definite form on \( V \). Instead define **Hermition inner product**
\[ \langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \]

to be s.t.

i) \[ \langle av_1 + bv_2, w \rangle = a \langle v_1, w \rangle + b \langle v_2, w \rangle \]
\[ \langle v, aw_1 + bw_2 \rangle = \bar{a} \langle v, w_1 \rangle + \bar{b} \langle v, w_2 \rangle \]

ii) \[ \langle v, w \rangle = \overline{\langle w, v \rangle} \]

iii) \[ \langle v, v \rangle \geq 0 \quad \text{equality} \iff v = 0 \]

\( \mathbb{C}^n \) has a standard hermitian inner product

\[ \langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i} = z_1 \overline{w_1} + \cdots + z_n \overline{w_n} \]

Then if \( T : \mathbb{C}^n \to \mathbb{C}^n \) linear, its Hermitian adjoint is \( T^* : \mathbb{C}^n \to \mathbb{C}^n \) defined by

\[ \langle Tv, w \rangle = \langle v, T^* w \rangle \]

If \( A \) is the matrix of \( T \), then the matrix of \( T^* \) is \( A^T = \overline{A} \).
Theorem (Spectral Theorem over $\mathbb{C}$) $T: \mathbb{C}^n \to \mathbb{C}^m$ is normal i.e. $TT^* = T^*T \iff T$ has an orthonormal basis of eigenvectors.

Note that Hermitian maps (i.e. $T$ s.t. $T = T^*$) are normal.

**Gram-Schmidt:** From a basis $v_1, \ldots, v_n$ we can obtain an orthonormal basis $e_1, \ldots, e_n$ by the procedure:

$$e_1 = \frac{v_1}{\|v_1\|}$$

$$e_2 = \frac{v_2 - \langle v_2, e_1 \rangle}{\|v_2 - \langle v_2, e_1 \rangle\|}$$

$$\vdots$$

$$e_n = \frac{v_n - \langle v_n, e_1 \rangle - \cdots - \langle v_n, e_{n-1} \rangle}{\|v_n - \langle v_n, e_1 \rangle - \cdots - \langle v_n, e_{n-1} \rangle\|}$$