Linear Algebra 1

Outline

Fundamentals

- Linear independence, bases, dimension
- Rank-Nullity
- Eigenvalues/diagonalizability
- Inner products and spectral theorem

Determinants

- Cofactor expansion
- Multilinear/alternating
- Triangular case

Characteristic and Minimal Polynomial

- Cayley-Hamilton
- Trace and determinant from characteristic polynomial

Problems

7.1.1 Let p, q, r, s be polynomials of degree at most 3. Which, if any, of the following two conditions is sufficient for the conclusion that the polynomials are linearly dependent?

1. At 1, each of the polynomials has value 0.
2. At 0 each of the polynomials has value 1.

7.2.7. Let T be a real symmetric $n \times n$ matrix with upper and lower diagonal having entries $b_1, ..., b_{n-1}$, and diagonal entries $a_1, ..., a_n$, all other entries zero (it is a *tridiagonal matrix*). Assume $b_j \neq 0$ for all j. Show

1. The rank of T is $\geq n - 1$ and
2. T has n distinct eigenvalues.

7.2.11. Let $\mathbb{R}[x_1, ..., x_n]$ be the polynomial ring over the real field $\mathbb{R}$ in the n variables $x_1, ..., x_n$. Let the matrix A be the $n \times n$ matrix whose ith row is $(1, x_i, x_i^2, ..., x_i^{n-1})$, $i = 1, ..., n$. Show that

$$det(A) = \Pi_{i>j}(x_i - x_j)$$

7.2.12. Consider an $(n+1) \times (n+1)$ matrix such that the ith row is $(1, a_i, a_i^2, ..., a_i^n)$, $i = 0, ..., n$ where the a_i are complex numbers .

1. Prove that this matrix is invertible if the a_i are all different. (Do not use 7.2.11).
2. In this case, prove that for any n complex numbers $b_0, ..., b_n$, there exists a unique complex polynomial f of degree n such that $f(a_i) = b_i$ for $i = 0, ..., n$.

7.5.13. Let F be a field, n and m positive integers, and A and $n \times n$ matrix with entries in F such that $A^m = 0$. Prove that $A^n = 0$.

7.6.5. Compute A^{10} for the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$