Outline

Basics
- Checking group axioms
- Working with generators
- Key examples (quaternion, dihedral, cyclic, symmetric, alternating; major matrix groups)
- Isomorphism theorems

Special properties of cyclic groups (F00 16)
- Characterization by subgroups
- Automorphism groups

Special properties of the symmetric group (F05 8A)
- Transposition decompositions, the alternating group
- Disjoint cycle decompositions, conjugacy
- Generating subsets (adjacent transpositions, n-cycle and one transposition)

Finitely-generated abelian groups
- Classification (two descriptions of torsion part)

Direct and semidirect products (F04 9B)
- Construction of semidirect products
- Recognition theorems

Group actions (F03 7B, S08 3B)
- Orbit-stabilizer theorem - useful for many counting problems
- Class equation
- $G$ acts on itself or a collection of its subgroups by conjugation. $G$ acts on the cosets of a subgroup by left (or right) multiplication.
- Think of as a map into $S_n$. Look at image and kernel.

Sylow theorems ($\star$)
- Argue by size (be careful about sizes of intersections)
- Have $G$ act on its Sylow subgroups by conjugation
- Cauchy theorem

Problems

Fall 2000 16  (Half of “characterization by subgroups”) Let $G$ be a finite group of order $n$ with the property that for each divisor $d$ of $n$ there is at most one subgroup in $G$ of order $d$. Show $G$ is cyclic.
Fall 2003 7B  (a) Let $G$ be a finite group and let $X$ be the set of pairs of commuting elements of $G$

$$X = \{(g, h) \subseteq G \times G : gh = hg\}.$$

Prove that $|X| = c|G|$ where $c$ is the number of conjugacy classes in $G$.

(b) Compute the number of pairs of commuting permutations on five letters.

Fall 2004 9B  Prove that every group of order 30 has a cyclic subgroup of order 15.

Fall 2005 8A  Find the smallest $n$ for which the permutation group $S_n$ contains a cyclic subgroup of order 111.

Spring 2008 3B  (“Poincare’s Theorem”) Let $G$ be a group and $H \leq G$ a subgroup of finite index $n$. Show that $G$ contains a normal subgroup $N$ such that $N \leq H$ and the index of $N$ is $\leq n!$.

Spring 2009 8B  1. Let $G$ be a non-abelian finite group. Show that $G/Z(G)$ is not cyclic, where $Z(G)$ is the center of $G$.
2. If $|G| = p^n$, with $p$ prime and $n > 0$, show that $Z(G)$ is not trivial.
3. If $|G| = p^2$, show that $G$ is abelian.

(*) Use the simplicity of $A_6$ to show that $A_6$ does not have an index 3 subgroup. Then show that there are no simple groups of order 120.