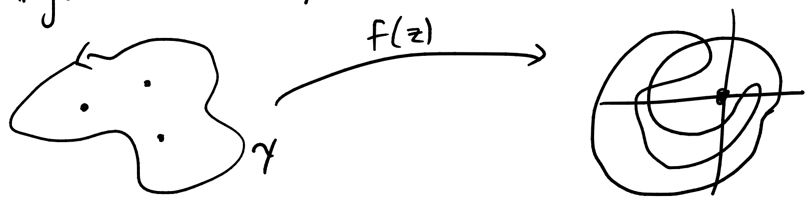
Complex II

Wednesday, August 9, 2023

Argument Principle



$\#\text{zeros} - \#\text{poles} = 1$ winds arounds once counterclockwise

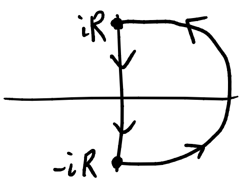
Net change in argument of $f(z)$ as z traverses γ $= 2\pi(\underbrace{\#\text{zeros} - \#\text{poles}}_{\text{inside } \gamma})$

$$f(z) = z^{2n} + \alpha^2 z^{2n-1} + \beta^2 = 0$$

n natural number ≥ 1

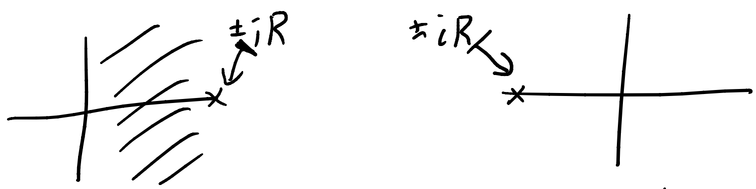
α, β real and nonzero

Show # roots with $Re(z) > 0$ $= \begin{cases} n & n \text{ even} \\ n-1 & n \text{ odd} \end{cases}$



$$f(iR) = (iR)^{2n} + \alpha^2 (iR)^{2n-1} + \beta^2$$
$$= (-1)^n R^{2n} + \cdots$$
$$f(-iR) = (iR)^{2n} + \cdots = (-1)^n R^{2n} + \cdots$$

n even $\qquad$ n odd



On the semicircle $|z| = R$, the z^{2n} term will dominate, and net change in argument will be $\approx 2n\pi$

$$f(it) = (it)^{2n} + \alpha^2 (it)^{2n-1} + \beta^2$$
$$= (-1)^n t^{2n} + (-1)^{n-1} i \alpha^2 t^{2n-1} + \beta^2$$

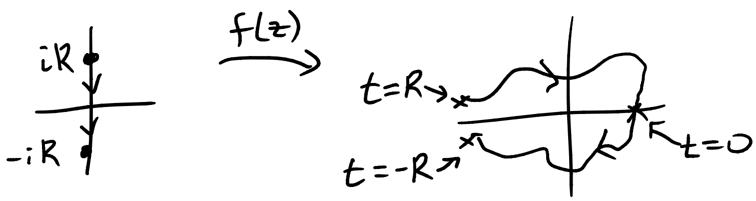
n even: $t^{2n} - i\alpha^2 t^{2n-1} + \beta^2$ stays in the right half plane $Re(z) > 0$, so no net change in argument.

Total net change in argument $= 2n\pi \implies n$ zeros.

n odd: $-t^{2n} + i\alpha^2 t^{2n-1} + \beta^2$

n odd: $-t^{2n} + i\alpha^2 t^{2n-1} + \beta^2$

When $t>0$, $\mathrm{Im}(f(z))>0$
When $t=0$, $f(z)=\beta^2$
When $t<0$, $\mathrm{Im}(f(z))<0$



Net change in argument is -2π
Total net change in argument $= 2n\pi - 2\pi \Rightarrow n-1$ zeros.

Rouche's Theorem

If $|f|<|g|$ on $\partial\Omega$
then g and $g+f$ have the same number of zeros minus poles.

Ex: How many zeros does
$$p(z) = z^5 + z^3 + 5z^2 + 2$$
have in $1<|z|<2$ (counting multiplicity)?

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When $|z|=1$, the $5z^2$ term dominates
$$|z^5+z^3+2| \leq 1+1+2 = 4 < |5z^2|$$
So Rouche says that $5z^2$ and $p(z)$ have the same number of zeros in $|z|<1$.

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When $|z|=2$, the z^5 term dominates
$$|z^3+5z^2+2| \leq 8+20+2 = 30 < 32 = |z^5|$$
So Rouche says that z^5 and $p(z)$ have the same number of zeros in $|z|<2$.

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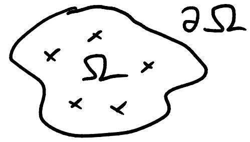
Residue Theorem

$$\int f(z)dz = \sum_i 2\pi i \, \mathrm{Res}[f(z), z=z_i]$$

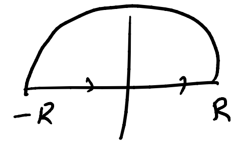


$$\int_{\partial\Omega} f(z)\,dz = \sum_{z_i} 2\pi i \,\underbrace{\text{Res}[f(z), z=z_i]}$$

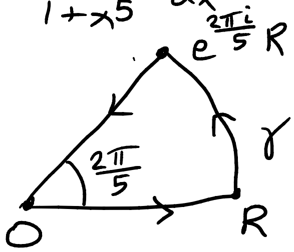
$$f(z) = \sum_k a_k (z-z_i)^k$$

Res at $z_i = a_{-1}$

Ex: Evaluate $\displaystyle\int_0^\infty \frac{1}{1+x^5}\,dx$



Key contour:



$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)$$

$$\int_\gamma f(z)\,dz = 2\pi i \sum \text{Res}$$

$z = t$
$0 \le t \le R$
$$\int_0^R \frac{1}{1+z^5}\,dz = \int_0^R \frac{1}{1+t^5}\,dt$$

$z = e^{\frac{2\pi i}{5}}t$
$0 \le t \le R$
$$\int_{e^{\frac{2\pi i}{5}}R}^0 \frac{1}{1+z^5}\,dz = \int_R^0 \frac{1}{1+(e^{\frac{2\pi i}{5}}t)^5}\, e^{\frac{2\pi i}{5}}\,dt$$

$$= -e^{\frac{2\pi i}{5}} \int_0^R \frac{1}{1+t^5}\,dt$$

$$\left|\int_\gamma \frac{1}{1+z^5}\,dz\right| \le \underset{\substack{\frac{2\pi R}{5}\\ O(R)}}{\text{length}} \cdot \underset{\substack{\frac{1}{R^5-1}\\ O(R^{-5})}}{\max} \to 0 \quad \text{as } R \to \infty$$

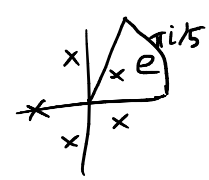
$$\left|\frac{1}{1+z^5}\right| = \frac{1}{|1+z^5|} \le \frac{1}{R^5-1}$$

$$R^5 = |z^5| = |z^5+1-1| \le |z^5+1| + 1$$

$$|z^5+1| \ge R^5-1$$

Where does $\frac{1}{z^5+1}$ have poles?

When $z^5 = -1$



$$\int_0^\infty \frac{dt}{1+t^5} - e^{\frac{2\pi i}{5}} \int_0^\infty \frac{dt}{1+t^5} = 2\pi i \,\text{Res}\left[\frac{1}{1+z^5}, e^{\pi i/5}\right].$$

$$\frac{1}{1+z^5} = a_{-1}(z-e^{\pi i/5})^{-1} + a_0 + \dots$$

$$\int^\infty \frac{dt}{} \quad 2\pi i \cdot \text{Res}$$

$$\int_0^\infty \frac{dt}{1+t^5} = \frac{2\pi i}{1-e^{2\pi i/5}} \cdot \text{Res}$$

$$\frac{1}{1+z^5} = a_{-1}(z-e^{\pi i/5})^{-1}+a_0+\dots$$

$$\frac{z-e^{\pi i/5}}{1+z^5} = a_{-1}+a_0(z-e^{\pi i/5})+\dots$$

$$= \frac{\frac{1}{5}e^{-4\pi i/5}}{1-e^{2\pi i/5}}2\pi i$$

$$\lim_{z\to e^{\pi i/5}} \frac{z-e^{\pi i/5}}{1+z^5} \overset{LH}{=} \lim_{z\to e^{\pi i/5}} \frac{1}{5z^4} = \frac{1}{5}e^{-4\pi i/5}$$

$$= \frac{\frac{1}{5}e^{-\pi i}}{e^{-\pi i/5}-e^{\pi i/5}}2\pi i$$

$$= \frac{+\frac{1}{5}2\pi i}{e^{\pi i/5}-e^{-\pi i/5}}$$

$$= \frac{\pi/5}{\frac{e^{\pi i/5}-e^{-\pi i/5}}{2i}} = \frac{\pi/5}{\sin(\frac{\pi}{5})} = \frac{\pi}{5\sin(\frac{\pi}{5})}$$

$0<a<b$

Evaluate $I = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|ae^{i\theta}-b|^4}d\theta$

Idea: Turn this into $\int_{|z|=1} f(z)\,dz = \int_0^{2\pi} f(e^{i\theta})ie^{i\theta}d\theta$

$z=e^{i\theta}$

$$I = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\left((ae^{i\theta}-b)\overline{(ae^{i\theta}-b)}\right)^2}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{((ae^{i\theta}-b)(ae^{-i\theta}-b))^2}d\theta$$

$$= \frac{1}{2\pi i}\int_0^{2\pi}\frac{ie^{2i\theta}}{((ae^{i\theta}-b)(a-be^{i\theta}))^2}d\theta$$

$$= \frac{1}{\ }\int \frac{z}{(az-b)^2(a-bz)^2}d\theta$$

pole at $\frac{a}{b}$

$$= \frac{1}{2\pi i} \int_{|z|=1} \frac{z}{(az-b)^2(a-bz)^2}$$

$$= \operatorname{Res}\left[\frac{z}{(az-b)^2(a-bz)^2},\ z=\frac{a}{b}\right]$$

$$\frac{z}{(az-b)^2} = \underbrace{\qquad\qquad}_{\text{value at } a/b} + \underbrace{\qquad\qquad}_{\text{derivative at } a/b}\left(z-\frac{a}{b}\right) + \cdots$$

$$\frac{1}{(a-bz)^2} = \frac{1}{b^2\left(z-\frac{a}{b}\right)^2}$$

Reside is times $\frac{1}{b^2}$

$$\cdots = \frac{a^2+b^2}{(a^2-b^2)^3} \quad \text{(lots of algebra)}$$