equivalent:

- Differentiable: $f'(z) = \lim_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$ exists
- Cauchy-Riemann: $f(x+iy) = u(x,y) + iv(x,y)$, $u_x = v_y$ and $u_y = -v_x$
- Holomorphic: Derivative exists and is continuous
- Analytic: Locally, $f(z)$ agrees with a power series
- Cauchy/Cauchy-Goursat/Morera: $\int_\Gamma f(z) = 0$

(Only need to check $\int_{\text{triangles}} f(z) = 0$ or $\int_{\text{squares}} f(z) = 0$)

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt \qquad \gamma : [a,b] \to \mathbb{C}$$

Prove that a uniform limit of complex analytic functions is complex analytic.

($f_n \to f$ unif means $\forall \varepsilon\ \exists n_0$ s.t. $|f_n - f| < \varepsilon$ for all $n \geq n_0$)

We must show that $\int_\Gamma f(z)\, dz = 0$

$$\int_\Gamma f(z)\, dz = \int_\Gamma \lim_{n\to\infty} f_n(z)\, dz = \lim_{n\to\infty} \int_\Gamma f_n(z)\, dz$$
$$= \lim_{n\to\infty} 0 = 0 \quad \blacksquare$$

$$\left| \int_\Gamma f(z)\, dz - \int_\Gamma f_n(z)\, dz \right|$$
$$\leq \text{length}(\Gamma) \cdot \sup |f_n - f| \to 0$$

A conformal map $f : U \to V$ is a bijective holomorphic function

$\Rightarrow f'(z) \neq 0$, angle preserving

Riemann Mapping Theorem $U \subsetneq \mathbb{C}$ $\underbrace{\text{simply connected}}^{\text{no holes}}$ and open,

Riemann Mapping Theorem

If $\emptyset \neq U \subsetneq \mathbb{C}$ $\overbrace{\text{simply connected}}^{\text{no holes}}$ and open, then U is conformally equivalent to the disc $\mathbb{D}$.

Möbius transforms / Linear Fractional transformations

$$z \mapsto \frac{az+b}{cz+d} : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \qquad \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

$\infty \mapsto \frac{a}{c}$

$-\frac{d}{c} \mapsto \infty$

- $\frac{az+b}{cz+d}$ behaves like $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

  $\frac{az+b}{cz+d} \circ \frac{a'z+b'}{c'z+d'}$ can be computed as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$

- take circles/lines to circles/line
- Möbius transformations are determined by $z_1 \mapsto w_1$, $z_2 \mapsto w_2$, $z_3 \mapsto w_3$

Ex $\mathbb{H} \to \mathbb{D}$

$\infty \mapsto i$

$i \mapsto 0$

$0 \mapsto -i$

$1 \mapsto 1$

$-1 \mapsto -1$

$$i\,\frac{z-i}{z+i}$$

$$\begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix}^{-1} = \dots$$

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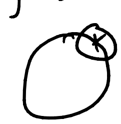
f analytic on $\mathbb{H}$, $|f| \leq 1$, $f(i) = 0$

How large can $f(2i)$ be?

Actually, $|f| < 1$ by the open mapping theorem



$2i \in \mathbb{H}$ corresponds to $i\,\frac{2i-i}{2i+i} = i\,\frac{i}{3i} = \frac{i}{3}$

$0 \longleftrightarrow i$

$$\mathbb{D} \longrightarrow \mathbb{H} \longrightarrow \mathbb{D}$$

to $i\frac{2i-i}{2i+i} = i\frac{i}{3i} = \frac{i}{3}$ $\quad \mathbb{D} \longrightarrow \mathbb{H} \longrightarrow \mathbb{D}$

New equivalent question: If f analytic $\mathbb{D} \to \mathbb{D}$, $f(0)=0$, how large can $f(\frac{i}{3})$ be?

$|f'(z)| \le 1$

Schwartz lemma: If $f: \mathbb{D} \to \mathbb{D}$, $f(0)=0$, then $|f(z)| \le |z|$, achieved by $f(z) = \lambda z$ with $|\lambda| = 1$.

Answer: $|f(\frac{i}{3})| \le \frac{1}{3}$.

Classification of conformal automorphisms $\mathbb{D} \to \mathbb{D}$

$$e^{i\theta}\,\frac{z-a}{1-\bar{a}z} \quad \text{for } a \in \mathbb{D}$$

$a \mapsto 0$

Classification ------- $\mathbb{C} \to \mathbb{C}$

just $az+b$ $(a \neq 0)$

Classification ------- $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$

just $\frac{az+b}{cz+d}$ with $ad-bc \neq 0$.

Any holomorphic map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is rational.

Singularities: f has an isolated singularity at z_0 if f is holomorphic on $B_\varepsilon(z_0) \setminus \{z_0\}$

- f has a Laurent series $\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$
- f has a removable singularity when $a_k = 0$ for all $k<0$.
- f has a pole when $a_k = 0$ for k sufficiently negative (i.e., only finitely many negative terms) (same as $f(z) \to \infty$ as $z \to z_0$)
- f has an essential singularity when $a_k \neq 0$ for infinitely many $k<0$.

(Casorati-Weierstrass: the image of $B_\varepsilon(z_0) \setminus \{z_0\}$ under f is dense in $\mathbb{C}$)

(Casorati-Weierstrass: the image of $B_\varepsilon(z_0)\setminus\{z_0\}$ under f is dense in $\mathbb{C}$)

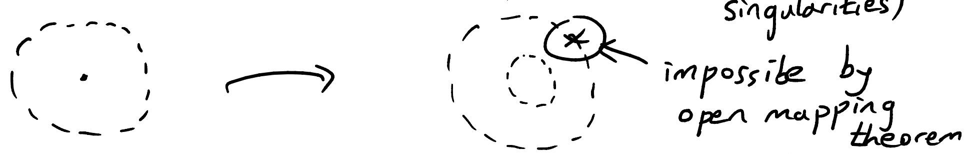
Show no conformal equivalence between $\{0<|z|<1\}$ and $\{1<|z|<2\}$

Suppose $f : \{0<|z|<1\} \to \{1<|z|<2\}$

f has an isolated singularity at 0.

Image of f is bounded $\Rightarrow$ Removable

(Riemann's theorem on removable singularities)



Where does 0 go? (after filling in the singularity)

A point near $f(0)$ will have 2 preimages, which is a contradiction.

Liouville's Theorem: A bounded entire function is constant.

(entire: analytic on all of $\mathbb{C}$)

Ex: If f, g entire, $\text{Re}(f) \le k\,\text{Re}(g)$, then $f = ag+b$.

Pf: $|e^z| = e^{\text{Re}(z)}$

$$\text{Re}(f) \le k\,\text{Re}(g)$$

$$e^{\text{Re}(f)} \le e^{\text{Re}(kg)}$$

$$|e^f| \le |e^{kg}|$$

$$|e^{f-kg}| \le 1.$$

⚠ You might worry that $\text{Re}(f) \le k\,\text{Re}(g) \Rightarrow \text{Re}(f) \le 2k\,\text{Re}(g)$ but this isn't an issue since the real parts could be negative.

So e^{f-kg} must be constant.

So $f - kg = c$

could

So $f - kg = c$

So $f = kg + c$.