Complex Analysis I

Outline

Holomorphic functions

- Cauchy-Goursat theorem
- Morera's theorem
- Holomorphic vs. real analytic functions

Conformal maps

- Riemann mapping theorem
- Important examples: disc to upper half-plane and vice versa, upper half-plane to a sector, Schwarz-Christoffel integral
- Automorphisms of the disk; Schwarz lemma, Blaschke factors

The extended/compactified complex plane

Classification of singularities

- Removable singularities
- Poles
- Essential singularities; Casorati-Weierstrass

Local and global behavior of holomorphic functions

- Open mapping theorem
- Maximum modulus principle
- Liouville's theorem

Problems

5.2.16 – Spring 1978 12 Prove that the uniform limit of a sequence of complex analytic functions is complex analytic. Is the analogous theorem true for real analytic functions?

5.4.13 – Fall 1990 13 Suppose that f is analytic on the open upper half-plane and satisfies $|f(z)| \leq 1$ for all z, $f(i) = 0$. How large can $|f(2i)|$ be under these conditions?

5.4.7 – Spring 2003 7B Let $f(z)$ be a function that is analytic in the unit disk $\mathbb{D} = \{|z| < 1\}$. Suppose that $|f(z)| \leq 1$ in $\mathbb{D}$. Prove that if $f(z)$ has at least two fixed points z_1 and z_2, then $f(z) = z$ for all $z \in \mathbb{D}$.

5.3.8 – Spring 1995 18 Prove that there is no one-to-one conformal map of the punctured disk $G = \{z \in \mathbb{C} : 0 < |z| < 1\}$ onto the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

5.5.3 – Fall 1999 4 Let the rational function f in the complex plane have no poles for

$\mathrm{Im}(z) \geq 0$. Prove that

$$\sup\{|f(z)| : \mathrm{Im}(z) \geq 0\} = \sup\{|f(z)| : \mathrm{Im}(z) = 0\}.$$

5.5.8 – Spring 1997 4 Let f and g be two entire functions such that, for all $z \in \mathbb{C}$, $\mathrm{Re}(f(z)) \leq k\mathrm{Re}(g(z))$ for some real constant k (independent of z). Show that there are constants a, b such that

$$f(z) = ag(z) + b.$$

5.6.29 – Spring 1987 15 Prove or disprove: If the function f is analytic in the entire complex plane, and if f maps every unbounded sequence to an unbounded sequence, then f is a polynomial.