Real Analysis

Outline

Continuous, real-valued functions on subsets of $\mathbb{R}$

- Intermediate value theorem, Mean value theorem
- Compactness properties

Sequences of real numbers

- lim sup and lim inf
- Monotone sequences

Other techniques

- Inequalities (Cauchy-Schwarz, Jensen's/convexity)
- Taylor's theorem with remainder

Multivariable versions

- Gradient, Hessian, Jacobian
- The implicit and inverse function theorems

Problems

1.1.10 - Fall 1982 11

1. Prove that there is no continuous map from the closed interval $[0, 1]$ onto (i.e. surjection) the open interval $(0, 1)$.
2. Find a continuous surjective map from the open interval $(0, 1)$ onto the closed interval $[0, 1]$.
3. Prove that no map in Part 2 can be bijective.

1.5.3 - Fall 1990 4 Suppose f is a continuous real valued function. Show that

$$\int_0^1 f(x)x^2dx = \frac{1}{3}f(\xi)$$

for some $\xi \in [0, 1]$.

1.3.8 - Spring 2003 6A Let x_n be a sequence of real numbers so that $\lim_{n\to\infty} 2x_{n+1} - x_n = x$. Show that $\lim_{n\to\infty} x_n = x$.

1.3.9 - Spring 2005 5 Let a and x_0 be positive numbers, and define the sequence $(x_n)_{n=1}^{\infty}$ recursively by

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right)$$

Show this sequence converges and find its limit.

1.5.9 - Fall 1985 15 Let $0 \le a \le 1$ be given. Determine all nonnegative continuous functions f on $[0,1]$ which satisfy the following three conditions:

$$\int_0^1 f(x)dx = 1, \int_0^1 xf(x)dx = a, \int_0^1 x^2 f(x)dx = a^2$$

1.5.9 - Fall 1985 15 Let f be a twice continuously differentiable function on the real line. Assume f is bounded with bounded second derivative. Let

$$A = \sup_{x\in\mathbb{R}} |f(x)|,\ B = \sup_{x\in\mathbb{R}} |f''(x)|.$$

Prove that

$$\sup_{x\in\mathbb{R}} |f'(x)| \le 2\sqrt{AB}$$

2.2.43 - Spring 1996 12 Let $M_{2\times 2}$ be the space of 2×2 matrices over $\mathbb{R}$, identified in the usual way with $\mathbb{R}^4$. Let the function F from $M_{2\times 2}$ to $M_{2\times 2}$ be defined

$$F(X) = X + X^2$$

Prove that the range of F contains a neighborhood of the origin.