Defn A metric space is a set X together with a function $d: X \times X \to \mathbb{R}$ satisfying the following:

i) $d(x,x) = 0$
ii) $d(x,y) = d(y,x)$
iii) $d(x,y) \leq d(x,z) + d(z,y)$

(e.g. $\mathbb{R}^n$ with $d(x,y) = |x-y|$)

A metric induces a topology on X. Denote

$$B_r(x) = \{ y \in X \mid d(x,y) < r \}$$

Then we say $\mathcal{U} \subseteq X$ is open if for all $x \in \mathcal{U}$ there is an $r_x > 0$ s.t. $B_{r_x}(x) \subseteq \mathcal{U}$.

A set $C \subseteq X$ is closed if C^c is open. Note a set can be clopen (e.g. X and ϕ are always clopen)

- Arbitrary unions of open sets are open
- Finite intersections of open sets are open

swap unions and intersections for closed sets

4.1.13 - Fall 1989 6 Take a point $z \in Y^c$. Take $\gamma = d(z,x) - r$. $\gamma > 0$ by definition of Y. Now consider $z' \in B_\gamma(z)$. Then $r < d(x,z) \leq d(x,z') + d(z',z)$

$$< d(x,z') + \gamma$$

$$\Rightarrow d(x,z') > r \text{ so } z' \in Y^c$$

Now we cover some more properties of metric spaces.

Defn A sequence $\{x_n\} \subseteq X$ converges to $x \in X$ if for any $\varepsilon > 0$ there is $N > 0$ s.t. $x_n \in B_\varepsilon(x)$ for all $n \geq N$.

E.g. $C \subseteq X$ is closed $\Leftrightarrow$ if $\{x_n\} \subseteq X$ converges to $x \in X$, then $x \in C$.

One property metric spaces have that more general topological spaces do not enjoy: For any $x, y \in X$ s.t. $x \neq y$ there exists open sets $x \in U_x$, $y \in U_y$ s.t.

$$\mathcal{U}_x \cap \mathcal{U}_y = \emptyset$$

called Hausdorff. One consequence of this is that a convergent sequence cannot converge to more than one element in X.

In metric spaces there is a sequential formulation of compactness. Here is the usual one:

Defn $C \subseteq X$ is compact if for every collection of open sets $\{\mathcal{U}_\alpha\}$ such that $C \subseteq \bigcup_\alpha \mathcal{U}_\alpha$, there exists a finite subset $\{\mathcal{U}_i\}$ s.t. $C \subseteq \bigcup_{i=1}^{n} \mathcal{U}_i$

"every open cover has a finite subcover"

Note for X a metric space, C compact $\Rightarrow$ C is closed and bounded (bounded meaning $d(c_1, c_2) < M$ $\forall c_1, c_2 \in C$)

The converse is usually not true (e.g. $(0, 1]$) except in $\mathbb{R}^n$ (Heine-Borel theorem)

Defn $C \subseteq X$ is <u>sequentially compact</u> if for every sequence $\{x_n\} \subseteq X$ there is a convergent subsequence

In the case of $\mathbb{R}^n$ this is usually called Bolzano–Weierstrass

For X a metric space

$$\text{Compactness} \iff \text{Sequential Compactness}$$

For X, Y metric spaces, the following definitions of a continuous function $f: X \to Y$ are equivalent

i) For each $x \in X$, for any $\varepsilon > 0$ there is a $\delta > 0$ s.t.

$$d(y, x) < \delta \Rightarrow d(f(y), f(x)) < \varepsilon$$

ii) For any $U \subseteq Y$ open, $f^{-1}(U)$ is open

iii) For any convergent sequence $\{x_n\} \subseteq X$
$x_n \to x$, then $f(x_n) \to f(x)$

Now, compactness is a "topological property", such properties should be preserved under continuous functions.

__Thm__ If $f: X \to Y$ continuous and $C \subseteq X$ is compact, then $f(C)$ is compact.

__pf__ Pull back open cover of $f(C)$ to C and then take an open subcover. $\square$ This gives extreme value Thm.

__4.2.6 - Fall 1980 19__ Suppose for contradiction there is $x \notin f(X)$. X is compact, thus $f(X)$ is also compact, and in particular closed. Then there is $\varepsilon > 0$ s.t. $d(x, f(X)) \geq \varepsilon$.

Now consider the sequence $\{f^n(x)\}$. Since X is a compact metric space, this should have a convergent subsequence. However, for any $m > n$

$$d(f^n(x), f^m(x)) = d(x, f^{m-n}(x)) \geq \varepsilon$$

and so there can't be a convergent subsequence

There is a slightly stronger version of continuity only available in metric space which sometimes comes up.

<u>Def</u> $f: X \to Y$ is <u>uniformly continuous</u> if for any $\varepsilon > 0$ there is $\delta > 0$ s.t.

$$d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$$

<u>Thm</u> If $f: X \to Y$ is continuous and X is compact, then f is uniformly continuous.

pf

$\varepsilon > 0$. For each $x \in X$ consider $\delta_x > 0$ s.t. $f(B_{\delta_x}(x)) \subseteq B_\varepsilon(f(x))$. Then take a subcover $B_{\delta_i}(x_i)$ and set $\delta = \min(\delta_i)$ $\square$

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X is compact and thus so is $f(X)$, thus f is bounded. Suppose $|f(x) - f(y)| < B$ for all x, y.

Now, as long as $|x-y| \geq c$ and $M \geq B/c$, then for any $\varepsilon > 0$

$$|f(x) - f(y)| \leq \varepsilon + M|x-y|$$

so just need to look at when $|x-y|$ is small. X is compact, thus f is uniformly continuous, thus

For $\varepsilon > 0$ there is $\delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever } |x-y| < \delta.$$

Thus set $M = B/\delta$ works.

We now consider $C(X) := \{ f : X \to \mathbb{R} \text{ continuous} \}$ where X is a $\boxed{\text{compact}}$ metric space.

Now this space is a vector space. Furthermore we can define a norm

$$\|f\| = \max |f|$$

i) $\|f\| \geq 0$, $= 0 \iff f \equiv 0$

ii) $\|cf\| = |c| \, \|f\|$

iii) $\|f+g\| \leq \|f\| + \|g\|$

These properties give that $d(f,g) = \|f-g\|$ forms $C(X)$ into a metric space.

Note if $f_n \to f$ in this metric that means

$$\max |f - f_n| \longrightarrow 0$$

which means f_n converges uniformly to f

$$\left(\text{i.e. } \forall\, \varepsilon > 0 \quad |f_n(\underset{\text{any}}{x}) - f(x)| < \varepsilon \text{ for large } n \right)$$

x

Recall, the result of a uniformly converging sequence of continuous functions is continuous (good practice to prove this)

$C(X)$ has the following special property:

<u>Def</u>:) If X is a metric space and $\{x_n\}$ a sequence s.t. for any $\varepsilon > 0$ there is $N > 0$ s.t.

$$d(x_{n_1}, x_{n_2}) < \varepsilon$$

for all $n_1, n_2 > N$, $\{x_n\}$ is called <u>Cauchy</u>.

ii) X is called <u>complete</u> if every Cauchy sequence is convergent

$\mathbb{R}^n$ is complete, and so $C(X)$ is as well.

Summary: $C(X)$ is a normed vector space, the metric induced by the norm is complete and convergence is uniform.

There is a characterization of compact sets in this space.

<u>Defn</u> A subset $\mathcal{F} \subseteq C(X)$ is <u>equicontinuous</u> if for each $x \in X$, for every $\varepsilon > 0$ there is $\delta > 0$ s.t.

$$|f(x') - f(x)| < \varepsilon$$

for all $d(x', x) < \delta$, for all $f \in \mathcal{F}$.

Uniform equicontinuity and equicontinuity equivalent when X is compact.

<u>Thm</u> If $\{f_n\} \subseteq C(X)$ a uniformly bounded equicontinuous sequence, then f_n has a uniformly convergent subsequence. Further, $\mathcal{F} \subseteq C(X)$ is s.t. $\overline{\mathcal{F}}$ is compact $\iff$ $\mathcal{F}$ is uniformly bounded and equicontinuous.

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Let $\varepsilon > 0$, $x \in X$. There is $f_1 \in \mathcal{F}$ s.t.

$$f_1(x) \leq g(x) \leq f_1(x) + \varepsilon$$

Now, let $\delta > 0$ s.t. if $y \in B_\delta(x)$ then

$$|f(x) - f(y)| < \varepsilon \quad \forall f \in \mathcal{F}$$

Then for $y \in B_\delta(x)$ there is $f_2 \in \mathcal{F}$ s.t.

$$f_2(y) \leq g(y) \leq f_2(y) + \varepsilon$$

Now, by definition of g,

$$f_2(x) \leq f_1(x) + \varepsilon$$

and so by equicontinuity f_2 can increase by at most ε, f_1 can decrease by at most ε, so

$$f_2(y) \le f_1(y) + 3\varepsilon$$

thus

$$f_1(y) \le g(y) \le f_1(y) + 3\varepsilon$$

and thus

$$|g(y) - g(x)| \le 2\varepsilon$$

for all $y \in B_\delta(x)$.

<u>Thm</u> (Banach fixed-point) Suppose X is a complete metric space and $T: X \to X$ is a map s.t. there is $q \in [0,1)$ with

$$d(T(x), T(y)) \le q\, d(x,y)$$

for all $x, y \in X$. Then there is a <u>unique</u> $x^* \in X$ s.t. $T(x^*) = x^*$.

Used for proving existence to differential equations, main idea in proof of Picard-Lindelof.

Fall 1982 18 Consider the map

$$T : C([0,1]) \longrightarrow C([0,1])$$
$$f \longmapsto e^{x^2} - \int_0^1 K(x,y) f(y)\,dy$$

Looking for a fixed point. We have

$$\|Tf - Tg\| = \max \left| \int_0^1 K(x,y)\big(g(y) - f(y)\big)\,dy \right| (*)$$
$$\leq \max |K(x,y)|\,|g(y) - f(y)|$$
$$\leq \big(\max |K(x,y)|\big)\ \|f - g\|$$

by assumption $\max |K(x,y)| < 1$, thus done

Last technique to mention is diagonalization.

Sometimes if I am trying to prove something about a sequence $\{x_n\}$ it is good to construct subsequences

$$\begin{array}{lll} x_{1,1}, & x_{1,2}, & \cdots \\ x_{2,1}, & x_{2,2}, & \cdots \\ x_{3,1}, & \cdots & \\ \vdots & & \ddots \end{array}$$

that have properties closer and closer to we want. Then choosing $\{x_{i,i}\}$ may have the desired property. For a great example, read the proof of Arzela-Ascoli.